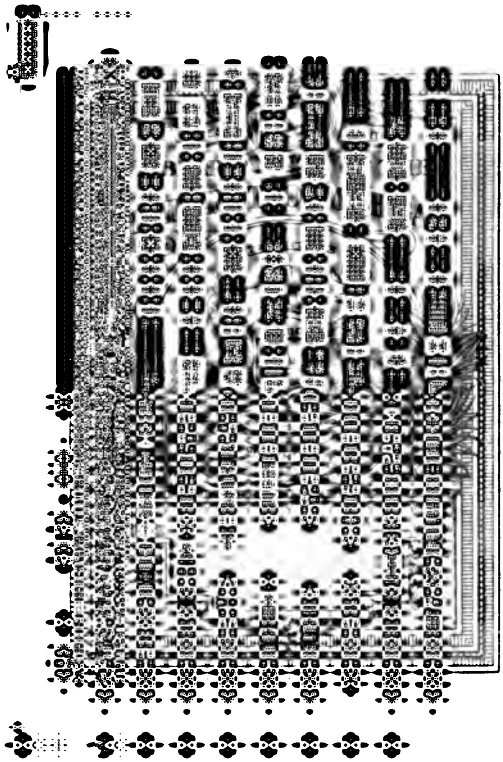

RY;

EMS, &c.

D.

# AN APPENDIX

TO THE LARGER EDITION OF

# EUCLID'S ELEMENTS OF GEOMETRY;

CONTAINING

ADDITIONAL NOTES ON THE ELEMENTS,
A SHORT TRACT ON TRANSVERSALS,
AND HINTS FOR THE SOLUTION OF THE PROBLEMS, &c.

BY

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THIS Appendix, presented to the student, is intended to supply some omissions in the notes, and to afford some assistance in the application of the principles of Geometry to the solution of Problems.

The first part supplies some additional notes on the elements, and a more full exposition of the method of the Geometrical Analysis.

The second consists of a short tract on the theory of Transversals, embracing only the most elementary properties.

The third part consists of hints, remarks, &c. on the Problems and Theorems. The constructive part of the analysis or synthesis is generally given, either wholly or partially, and the rest with the demonstration is left to the student. In some instances only brief remarks and references to Euclid are given.

These hints and remarks, it is hoped, will be found serviceable to the student in his geometrical studies, without paralyzing the efforts of his own mind.

R. P.

TRINITY COLLEGE,
*17th November*, 1847.

## ERRATUM.

In page 9, line 27, after the words "to construct the triangle," add "which shall have its vertex in the circumference of the given circle."

## ADDITIONAL NOTES TO BOOK I.

It has been maintained by some writers both of ancient and modern times, that Geometry is a perfectly abstract Science, a body of truths completely independent of all human observation and experience. The truths of Geometry may, possibly, be a portion of absolute and universal truths, such as change not in all time, and which maintain the same constant universality under every conceivable state of existence. Still it must be admitted that however abstract and independent of experience such truths may be in themselves, it was not as such that they were originally discovered, or in that form that they are apprehended by the human mind.

The natural process of the human mind in the acquirement of knowledge and in the discovery of truth, is, to proceed from the *particular* to the *general*, from the *sensible* to the *abstract*. It is perhaps not too much to affirm, that the human mind would never have speculated on the abstract properties of circles and triangles unless some *visible forms* of such figures had first been exhibited to the senses. It does seem more probable and analogous to the rise and progress of other branches of human knowledge, that the fundamental truths of Geometry should have been first discovered from suggestions made to the senses; and this opinion, too, is not repugnant to the earliest historical notices existing on the subject. The human mind is so constituted that exact knowledge in any Science can only be acquired progressively. Every successive step in advance must be taken as a sequel to, and dependent upon, the previous acquirements; and some intelligible facts and first principles must form the basis of all human Science.

Now the only possible way of explaining terms denoting simple perceptions is to excite those simple perceptions. The impossibility of defining a word expressive of a simple perception is well known to every one who has paid any attention to his own intellectual progress. The only way of rendering a simple term intelligible is to exhibit the object of which it is the sign, or some sensible representation of it. A straight line therefore must be drawn, and by drawing a curved line and a crooked line, the distinction will be perfectly understood. Again, the definition of a complex term consists merely in the enumeration of the simple ideas for which it stands, and it will be found that all definitions must have some term or terms equally requiring definition or explanation with the one defined. The sensible evidence of things is only to be acquired by the evidence of the senses.

The definitions of Euclid appeal directly to the senses; and the fundamental theorem (Euc. I. 4) which forms the basis of all the succeeding propositions, is demonstrated by one of the simplest appeals to experience. At every step there is a reference made to something exhibited to the senses, the coincidence of the lines, the angles, and lastly, the surfaces of the two triangles, and by shewing a perfect coincidence, their equality is inferred. The instance exhibited, and the proof applied to it, is equally valid for any triangles whatever which have the same specified conditions given in the hypothesis. The same reasoning may be applied to any similar case which can be conceived, and thus from a single instance demonstrated by appeal to the senses, we are led to admit the statement contained in the general enunciation. These considerations appear to support the opinion, that the truths of Geometry, as a portion of human Science, rest ultimately on the evidence of the senses.

It may also be suggested, whether it be not a point of considerable importance to be able to discriminate, where human Science begins, and how certainty is acquired.

Prop. XXIX. With respect to the different proposals made for the amendment of Euclid's method of treating the subject of parallel straight lines, it may be observed, that they all consist in setting out either with a different or a modified result from that of Euclid,—all true, and more or less obvious to the senses.

Euclid has discussed the elementary properties of *triangles*, or of two lines which meet one another and are intersected by a third line, before he has entered upon the discussion of the properties of two lines which do not meet when they are intersected by a third line. The principal objection to Euclid's method of treating the subject of parallel lines, is the assumption of one truth as an *axiom*, which forms the converse of a *theorem* which he has demonstrated as the seventeenth proposition of the first book. Almost every writer on the subject admits the necessity of assuming some positive property of parallel lines as the basis of the reasonings on such lines: and that amendment of Euclid's method would seem to be the best which simply supplies a defect, and leaves the so-called twelfth axiom to assume its rightful position as a theorem, and to fall into its proper place after the seventeenth proposition.

Two straight lines in the same plane which do not meet, when produced, may be *convergent* or *divergent* with respect to each other, according to the directions in which both lines are produced; or, when produced in either direction, they may be neither *divergent* nor *convergent*.

When a third line falls upon two straight lines and makes the two interior angles on one side of it less than two right angles; on that side of the line, the two straight lines are *convergent*, and will, if produced far enough, meet one another, as it is stated in the so-called twelfth axiom. On the other side of the line, the two interior angles are greater than two right angles, and the two straight lines are *divergent*, and will never meet, how far soever they may be produced.

The limiting position of the two straight lines, is, when they are neither *convergent* nor *divergent*, that is, when they do not meet when produced in either direction; and such lines are then said to be *parallel* to one another. If the two parallel lines be intersected by a third line, the following properties exist respecting the angles, whether the intersecting line be perpendicular or be not perpendicular to either of the parallel lines.

(1) The two interior angles on each side of the intersecting line are equal to two right angles. Euc. I. 28.

(2) The alternate angles on each side of the intersecting line are equal to one another. Euc. I. 27.

(3) The exterior angles are equal to their corresponding interior angles on the same side of the intersecting line. Euc. I. 28.

If the intersecting line be perpendicular to one of the parallel lines; it is also perpendicular to the other: and

(4) The perpendicular distance between the two lines is always the same.

If it has been correctly stated, that all *axioms* are in reality *theorems* assumed without proof, and that all demonstrated truths must depend on some truths assumed or admitted to be true, not necessarily truths first discovered, but truths the most simple and which arise directly from the subject of the definitions; the doctrine of parallel lines may be legitimately treated by assuming some one of the positive properties of such lines as the basis for demonstrating their other properties.

Any one of the four positive properties just stated may be assumed as the foundation of the theory of parallel lines, and that theory may be made to depend on the *distance* between the parallel lines, or on some of the angles made by any intersecting line. If the former assumption be adopted: does it not involve that lines which are perpendicular to one of the parallel lines, are also perpendicular to the other, as well as that all such distances are equal? This would require more to be taken for granted, than

would be necessary by assuming the equality of the exterior and interior angles, or either of the two remaining properties respecting the angles which the parallels make with any intersecting line.

In general, with respect to *indirect* demonstrations, it may be questioned whether they ought ever to be admitted as a legitimate mode of proof in a primary and fundamental proposition. *Indirect* demonstrations are properly and most effectually applied in proving the converse of a proposition which has been demonstrated by a *direct* appeal to assumed or demonstrated principles. To make a negative property or an indirect demonstration the basis of a *positive* doctrine, seems to be an inversion of the natural process the mind pursues in the investigation of truth, and to leave the doctrine exposed to all the objections which may be made from the illogical attempt to prove a *positive* from a *negative* truth. See De Morgan's Connexion of Numb. & Magnitude p. 51.

For a more philosophical view of the subject, reference may be made to Professor Powell's able Pamphlet, "On the Theory of Parallel Lines."

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## ADDITIONAL NOTES TO BOOK V.

The doctrine of Ratio and Proportion is one of the most important in the whole course of mathematical truths, and it appears probable that if the subject were read simultaneously in the Algebraical and Geometrical form, the investigations of the properties, under both aspects, would mutually assist each other, and both become equally comprehensible; also their distinct characters would be more easily perceived.

In the definition of Ratio as given by Euclid, all reference to a third magnitude of the same geometrical species, as a measure for comparing the two, whose ratio is the subject of conception, has been carefully avoided. It is their relation one to the other, without the intervention of their sum, their difference, or any standard unit whatever. One of the magnitudes is made the standard by which the other is estimated; but even this is not effected by means which require the inquiry, "how many times is the one contained in the other?" Such a procedure would, undoubtedly, have been legitimate, had it been also convenient: but it would at once have led to considerations respecting fractions or irrational functions. Euclid effects his demonstrations by the aid of *multiples* instead of *quotients*: by repetitions of the magnitudes themselves, instead of finding what multiple the one magnitude is of the other, or what multiple each of them is of some third magnitude. Euclid's results too are obtained with greater logical brevity, and he employs fewer principles in their establishment, than any writer has yet been able to do, *in a strictly legitimate form*, by means of the Modern Algebra.

The simple idea of ratio of itself, and absolutely considered, could not, however, lead to any conclusion respecting the properties of figures any more than the mere idea of magnitude. *It is by the comparison of two or more magnitudes subjected to some specific conditions* in the first four books of the Elements, that all the propositions have been demonstrated: and *it is by the comparison of the ratios of two or more pairs of correlative magnitudes, subject to specified conditions,* that the properties of figures depending on ratio are to be established. Euclid does not offer even a solitary property of a single ratio, or of the magnitudes whose ratio it is: except, indeed, that already adverted to, as constituting, in fact, an axiom.

As each of two ratios involves the idea of two magnitudes, the least number of magnitudes between which a comparison of ratios is possible, is four, two for each of the ratios. When these ratios are equal, the sixth definition gives the name of *proportionals* to them collectively, and points out the mode in which they are to be spoken of, and the ordinary, though somewhat inconvenient mode of writing them.

The fifth definition is that of equal ratios. The definition of ratio itself (defs. 3, 4) contains no criterion by which one ratio may be known to be equal to another ratio; analogous to that by which one magnitude is known to be equal to another magnitude (Euc. I. Ax. 8). The preceding definitions (3, 4) only restrict the conception of ratio within certain limits, but lay down no test for comparison, or the deduction of properties. All Euclid's reasonings were to turn upon this comparison of ratios, and hence it was competent to lay down a criterion of equality and inequality of two ratios between two pairs of magnitudes. In short, his *effective* definition is a definition of proportionals.

The precision with which this definition is expressed, considering the number of conditions involved in it, is remarkable. Like all complete definitions, the terms (the subject and predicate) are convertible: that is,

(*a*) If the four magnitudes be proportionals, and any equimultiples be taken as prescribed, they shall have the specified relations with respect to "greater, greater, &c."

(*b*) If of four magnitudes, two and two of the same Geometrical Species, it can be shewn that the prescribed equimultiples being taken, the conditions under which those magnitudes exist, *must be* such as to fulfil the criterion "greater, greater, &c."; then these four magnitudes shall be proportionals.

It may be remarked, that the cases in which the second part of the criterion ("equal, equal") can be fulfilled, are comparatively few: namely, those in which the given magnitudes, whose ratio is under consideration, are both exact multiples of some third magnitude—or those which are called *commensurable*. When this, however, is fulfilled, the other two will be fulfilled *as a consequence of this*. When this is not the case, or the magnitudes are *incommensurable*, the other two criteria determine the proportionality. However, when no hypothesis respecting commensurability is involved, the contemporaneous existence of the three cases ("greater, greater; equal, equal; less, less") must be deduced from the hypothetical conditions under which the magnitudes exist, to render the criterion valid.

The following *axioms*, though not expressed by Euclid, are virtually employed by him, and may be added to the four he has given.

Ax. 5. A part of a greater magnitude is greater than the same part of a less magnitude.

Ax. 6. That magnitude of which any part is greater than the same part of another, is greater than that other magnitude.

The fifth book of the Elements as a portion of Euclid's System of Geometry ought to be retained, as the doctrine contains some of the most important characteristics of an effective instrument of intellectual Education. This opinion is favoured by Dr Barrow in the following expressive terms: "There is nothing in the whole body of the Elements of a more subtile invention, nothing more solidly established, or more accurately handled than the doctrine of proportionals."

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## ADDITIONAL NOTES TO BOOK VI.

Prop. XXIII. The doctrine of compound ratio, including duplicate and triplicate ratio, in the form in which it was propounded and practised by the ancient Geometers has been almost wholly superseded. However satisfactory for the purposes of exact reasoning the method of expressing the ratio of two surfaces, or of two solids by two straight lines, may be in itself, it has not been found to be the form best suited for the direct application of the results of Geometry. Almost all modern writers on Geometry and its applications to every branch of the Mathematical Sciences, have adopted

the algebraical notation of a quotient AB : BC; or of a fraction $\frac{AB}{BC}$; for expressing the ratio of two magnitudes: as well as the form of a product $AB \times BC$, or $AB . BC$, for the expression of a rectangle. The want of a concise and expressive method of notation to indicate the proportion of Geometrical Magnitudes in a form suited for the direct application of the results, has doubtless favoured the introduction of Algebraical symbols into the language of Geometry. It must be admitted, however, that such notations in the language of pure Geometry are liable to very serious objections, chiefly on the ground that pure Geometry does not admit the Arithmetical or Algebraical idea of a *product* or a *quotient* into its reasonings. On the other hand, it may be urged, that it is not the employment of symbols which renders a process of reasoning peculiarly Geometrical or Algebraical, but the ideas which are expressed by them. If symbols be employed in Geometrical reasonings, and be understood to express the *magnitudes themselves* and the *conception of their Geometrical ratio*, and not any *measures*, or *numerical values of them*, there would not appear to be any very great objections to their use, provided that the notations employed were such as are not likely to lead to misconception. It is, however, desirable, for the sake of avoiding confusion of ideas in reasoning on the properties of number and of magnitude, that the language and notations employed both in Geometry and Algebra should be rigidly defined and strictly adhered to, in all cases. At the commencement of his Geometrical studies, the student is recommended not to employ the symbols of Algebra in Geometrical demonstrations (see preface). How far it may be necessary or advisable to employ them when he fully understands the nature of the subject, is a question on which some difference of opinion exists.

The following is an example of the method which is generally used:

If (figure Euc. VI. 23) the parallelograms be supposed to be rectangular.

Then the rectangle AC : the rectangle DG :: BC : CG, Euc. VI. 1.

and the rectangle DG : the rectangle CF :: CD : EC,

whence the rectangle AC : the rectangle CF :: BC . CD : EC . CG.

Or, the areas of two rectangles are proportional to the products of the units contained in their adjacent sides respectively.

If, however, we agree that the ratio of BC . CD to EC . CG shall be interpreted to represent the ratio compounded of the ratios of the adjacent sides of the rectangles; we may express the proportion in the following form.

The ratio of the areas of two rectangles is as the ratio which is compounded of the ratios of their bases and altitudes.

Also, in a similar way, it may be shewn that the ratio of the areas of any two parallelograms, whether they be equiangular or not, is as the ratio compounded of the ratios of their bases and altitudes.

In conclusion, the Student must always remember that the introduction of numerical measures or Algebraical operations, must be regarded as a departure from the ancient Geometry, which, as a Science, recognizes no unit of measurement whatever.

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## ON THE CLASSIFICATION OF PROPOSITIONS.

THERE are only two forms of Propositions in the Elements, the *theorem* and the *problem*. In the *theorem*, it is asserted, and is to be proved, that if a geometrical figure be constructed with certain specified conditions, then some other specified relations must *necessarily* exist between the constituent parts of that figure. Thus:—if squares be described on the sides and hypothenuse of a right-angled triangle, the square on the

hypothenuse must *necessarily* be equal to the other two squares upon the sides (Euc. I. 47). In the *problem*, certain things are given in magnitude, position, or both, and it is required to find certain other things in magnitude, position, or both, that shall *necessarily* have a specified relation to the data, or to each other, or to both. Thus:—a circle being given, it may be required to construct a pentagon, which shall have its angular points in the circumference, and which shall also have both all its sides equal, and all its angles equal. (Euc. IV. 11.)

In Euclid's propositions, it may be remarked, there is in general, an aim at *definiteness*, considered in reference to the *quæsitum* of the problem, and the *predicate* of the *theorem*. The *quæsitum* of the problem is either a single thing, as the perpendicular in Euc. I. 11; or at most two, as the tangents to the circle in the first case of Euc. III. 17; and in the most general problems, even those which transcend the ordinary geometry, the solutions are always restricted to a definite number, which can always be assigned *a priori* for every problem. In certain cases, however, the conditions given in Euclid are not sufficient to fix entirely the quæsitum *in all respects*. For instance, Euc. IV. 10, the magnitude of the triangle is any whatever, and therefore not entirely fixed in all respects; or, again, in Euclid IV. 11, the pentagon may be any whatever, so that its position in the circle is not fixed. To fix the *magnitude* of the triangle or the *position* of the pentagon, some other condition independent of the data, must be added to the conditions of the problem. The length and position of some line connected with the triangle, (as one of the equal sides, the base, the perpendicular, &c.) would have fixed the triangle in magnitude and position; and the position of one angular point of the pentagon, or the condition that one side of the pentagon should pass through a given point (though this point must be subject to a certain restriction as to position, if within the circle), or any other possible conditions, would have confined the pentagon to a single position, or to the alternative of two positions. Such is the only kind of indeterminateness in the *problems* of "the Elements." In the enunciation of the *theorems* too, the same aim at *singleness* in the property asserted to be consequent on the hypothesis, is apparent throughout. There is, however, a remarkable difference in the characters of the hypotheses themselves, in Euclid's theorems: viz.

(1) That in some of them, one thing alone, or a certain definite number, possess the property which is affirmed in the enunciation.

(2) That all the things constituted subject to the hypothetical conditions, possess the affirmed property.

As instances of the first class, the greater number of theorems in the Elements may be referred to, as Euc. I. 4, 5, 6, 8, which are of the simplest class. In these, only one thing is asserted to be equal to another specified thing. In all the theorems of the Second Book, one thing is asserted to be equal to several other things taken together; and the same occurs in Euc. I. 47, as well as frequently in the other Books. They sometimes also take the form of asserting that no certain magnitude is greater or less than another, as in Euc. I. 16, or that two things together are less than or greater than, some one thing or several things, as Euc. I. 17. In all cases, however, this class is distinguished by the circumstance, that the things asserted to have the property are of a given finite number.

As instances of the second class, reference may be made to Euc. I. 35, 36, 37, 38, where all the parallelograms in the two former, and all the triangles in the two latter, are asserted to have the property of being equal to one given parallelogram or one given triangle. Or to Euc. III. 14, 20, 21; the lines in the circle in Prop. 14, or the angles at the circumference in Props. 20, 21, are any whatever, and therefore *all* the lines or angles constituted as in the enunciations, fulfil the conditions. Or again, in Book V. the two pairs of indefinite multiples, which form the basis of Euclid's definition of pro-

portionals; or his propositions "*ex æquo*," and "*ex æquo perturbato*," and the Propositions F, G, H, K; or, lastly, Euc. VI. 2, in which the property is (really, though not formally,) affirmed to be true when any line is drawn parallel to any one of the sides of the triangle.

The very circumstance, indeed, just noticed parenthetically, prevails so much in Euclid's enunciations, as to render it clear that it was his object as much as possible to render the conditions of the hypothesis *formally* definite in number; and if these remarks had no prospective reference, the circumstance would scarcely deserve notice. Still, with such prospective reference, it is necessary to insist upon the fact, that however the form of enunciation may be calculated to remove observation from it, the hypothesis itself is indefinite, or includes an indefinite number of things, which an additional condition would, as in the case of the problem, have restricted either to one thing or to a certain number of things.

Sometimes too, the theorem is enunciated in the form of a negation of possibility, as Euc. I. 7, III. 4, 5, 6, &c. These offer no occasion for remark, except to the ingenious modes of demonstration employed by Euclid. All such demonstrations must necessarily be indirect, assuming as an admitted truth the possibility of the fact denied in the enunciation.

Both among the Theorems and Problems, cases occur in which the hypotheses of the one, and the data or quæsita of the other, are restricted within certain limits as to *magnitude* and *position*. The determination of these limits constitutes the doctrine of *Maxima and Minima*. Thus:—the limit of possible diminution of the sum of the two sides of a triangle described upon a given base, is the magnitude of the base itself, Euc. I. 20, 22:—the limit of the side of a square which shall be equal to the rectangle of the two parts into which a given line may be divided, is half the line, as it appears from Euc. II. 5:—*the greatest line* that can be drawn from a given point within a circle, to the circumference, Euc. III. 7, is the line which passes through the centre of the circle; and *the least line* which can be so drawn from the same point, is the part produced, of the greatest line between the given point and the circumference. Euc. III. 8, also affords another instance of a maximum and a minimum when the given point is outside the given circle.

The theorem Euc. VI. 27 is a case of the *maximum* value which a figure fulfilling the other conditions can have; and the succeeding proposition is a problem involving this fact among the conditions as a part of the data, in truth, perfectly analogous to Euc. I. 20, 22; and finally, there are instances either direct or virtual in Euc. XI. 20, 21, 22, 23.

The doctrine itself was carefully cultivated by the Greek Geometers, and no solution of a Problem or demonstration of a theorem was considered to be complete, in which it was not determined, whether there existed such limitations to the possible magnitudes concerned in it, and how those limitations were to be actually determined.

Such Propositions as directly relate to *Maxima* and *Minima*, may be proposed either as Theorems or Problems. For the most part, however, it is the more general practice to propose them as Problems; but this has most probably arisen from the greater brevity of the enunciations in the form of a Problem. When proposed as a Problem, there is greater difficulty involved in the solution, as it required to find the limits with respect to *increase* and *decrease;* and then to prove the truth of the construction: whereas in the form of a Theorem, the construction itself is given in the hypothesis.

It may be remarked that though the Differential Calculus is always effective for the determination of Maxima and Minima, (in cases where such exist) yet in numerous cases, where it is applied to Problems of the classes which were cultivated by the Ancient Geometers, it is far less direct and elegant in its determinations than the Geometrical

methods. Now if reference be made to what has been stated respecting Theorems, where the hypothesis is *indeterminate*, or wanting in that completeness which reduces the property spoken of to a single example of the figure in question, a consequence of that *peculiarity* in such classes of Propositions may be remarked.

This peculiarity introduces another class of Propositions, which, though in "the Elements" somewhat disguised, formed an important portion of the Ancient Geometry :—the doctrine of *Loci*.

If the converse of Euc. I. 34, 35, 36, 37, and Euc. III. 20, 21, be taken in the form of Problems, they will become,

(1) Given the base and area, to construct the parallelogram.

(2) Given the base and area, to construct the triangle.

(3) Given the base and vertical angle, to construct the triangle.

Now *three conditions* are necessary to fix the magnitude of a triangle or a parallelogram, and in general, three only are sufficient for the purpose; but here it will be observed that only two are given in each case. The precise triangle or parallelogram, viewed as peculiarly solving the Problem, cannot be separated from all the others, except by adding some third condition to the two already given.

The side of the parallelogram in (1), and the vertex of the triangle in (2), opposite to the base, may be in any positions in a certain line parallel to the base; and the vertex of the triangle in (3), may be at any point in the circumference of a segment of a certain circle. The parallel line in which the vertices of all the equal triangles are situated, in one case, and the arc of the circle in which the vertices of all the triangles having equal vertical angles are situated, is each called the *locus of the vertex* of the triangle, since it occupies, in each case, *all the places* in which that vertex may be situated so as to fulfil the required conditions. In the same way, the parallel to the base is also the locus of all the positions in which the other two angular points of the parallelogram may be situated. These Problems are the simplest instances of that class which is called *Local Problems;* and their peculiar character is, that the data are one less than the number of conditions required by the nature of the Problem to restrict the quæsitum to a single or specified number of cases; as in these Problems the data consist of two conditions, while the exactly defining conditions must be three.

Again, viewed as Theorems, they may be thus enunciated :—

(1) If the base and area of a parallelogram be given, the locus of the other angular points will be a straight line parallel to the base.

(2) If the base and area of a triangle be given, the locus of its vertex is a straight line parallel to the base.

(3) If the base and vertical angle of a triangle be given, the locus of the vertex will be an arc of a circle.

In the original form of the propositions, the entire meaning, and that justified by Euclid's own reasoning, is that which would result from saying, "all triangles," "all parallelograms," &c. It will obviously be the case here, as in the *Maxima* and *Minima*, that the proposition may be enunciated either as a *local theorem* or as a *local problem;* and the circumstances will be similar as to the comparative brevity of enunciation and difficulty of the solution, when the proposition is given in the form of a Problem.

The great use made of *loci* by the Ancient Geometers was in the construction of determinate Problems. A certain number of data are required according to the nature of the Problem for rendering the quæsitum determinate; as, for instance, those in the case of the triangle. The subject will be better illustrated by an example, and one may be founded on the second and third propositions already noticed, which will take the following form :—

Given the base, the area, and the vertical angle of a triangle, to construct it.

When the base and the area of a triangle are given, the locus of its vertex is a straight line which can be determined from these data; and when the base and vertical angle are given, the locus of the vertex is a portion of the circumference of a circle which can be determined from these data. Now the point or points of intersection of these loci, will fulfil both conditions, that the triangle shall have the given area, and the given vertical angle. To express the principle generally:—let there be $n$ conditions requisite for the determination of a point which either constitutes the solution, or upon which the solution of the problem depends. Find the locus of this point subject to $(n-1)$ of these conditions; and again, the locus of the point subject to any other $(n-1)$ of these conditions. The intersection of these two loci gives the point required.

It may be observed that $(n-2)$ of the data must be the same in determining the two loci, and no one of the $n$ data must be a consequence of, or depend upon, the remaining $(n-1)$ data, in other words, the $n$ data must separately express $n$ independent conditions.

There are however cases in which one datum is involved in another, and these are of two different kinds—*essential* and *accidental*. To illustrate this distinction, let the following Problem be taken;

Given the base, the area, and the perpendicular drawn from the vertex to the base of the triangle, to construct it. Or,

Given the base, the vertical angle and the sum of the other two angles at the base of the triangle, to construct it.

Now in each of these problems, the third datum is absolutely determined and invariable, in consequence of its essential dependence on the two previous ones.

This dependence is universal and *essential*.

Again, suppose the problem were;—

Given the base of a triangle and a circle in magnitude and position, and likewise the vertical angle, to construct the triangle.

In this case, the given circle will generally be a different one from that which forms the locus of the vertical angle, and in that case, the intersections, or the point of contact, of the two circles will give either two solutions or one solution of the Problem. But on the other hand, the given circle *may* coincide with the locus, and thus again render the Problem indeterminate in this particular case. Generally the construction is possible, and only *accidentally* it becomes indeterminate.

The distinction between these two cases is very important.

As Problems are generally constructed by the intersections of loci, it is easy to imagine cases and conditions that shall give loci which can never meet.

For instance, in the problem just stated, the two circles may never meet; and in the preceding one, the straight line and circle may never meet. In all such cases the problem is *impossible* with the given conditions: these conditions being incompatible with each other in their nature, or more frequently, in their magnitude and position, and with the co-existence of that which constitutes the quæsitum. The limiting cases of possibility belong to the doctrine of Maxima and Minima.

The importance of the distinction alluded to, when one datum is contained in another, arises from its constituting the foundation of another Class of Propositions. These are the *Porisms*.

Whenever the quæsitum is a point, the problem on being rendered indeterminate, becomes a locus, whether the deficient datum be of the essential or of the accidental kind. When the quæsitum is a straight line or a circle, (which were the only two loci admitted into the ancient Elementary Geometry) the problem *may* admit of an *accidentally indeterminate* case; but will not *invariably* or even very frequently do so. This will be the case, when the line or circle shall be so far arbitrary in its position, as

depends upon the deficiency of *a single* condition to fix it perfectly:—that is, (for instance) one point in the line, or two points in the circle, may be determined from the given conditions, but the remaining one is indeterminate from the accidental relations among the data of the problem.

Determinate Problems become indeterminate by the merging of some one datum in the results of the remaining ones. This may arise in three different ways; first, from the coincidence of two points; secondly, from that of two straight lines; and thirdly, from that of two circles. These, further, are the only three ways in which this accidental coincidence of data can produce this indeterminateness; that is, in other words, convert the Problem into a Porism.

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## GEOMETRICAL ANALYSIS.

The term Analysis is usually understood to signify the separation of any thing into its constituent parts for the purpose of examining them separately; but as employed in Geometry, it expresses a reversal of the order of the parts of a demonstration, or an examination of the conditions attached to the construction of a Problem. The term used in its strict etymological sense is not exactly in accordance with the use made of it in Geometry. Yet this is of little importance, so that its applied meaning be clearly understood:—a meaning which it is difficult to express in the same words when applied to the *Theorem* and the *Problem*. For the purpose of securing perspicuity, it is therefore deemed the better plan, to consider it separately under the aspect it frequently bears in each of these applications. The very descriptive use of it given by Leslie, in his usually forcible antithetic manner, will in each case, be very striking: "Analysis presents the medium of invention; while Synthesis naturally directs the course of instruction."

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## THE ANALYSIS OF THEOREMS.

It will have been remarked, that in the Elements, Euclid frequently uses the *indirect* method of demonstration:—that is, of proving the truth of a theorem by demonstrating that a contrary conclusion is incompatible with the hypothesis of that theorem. To effect this, he *supposes* the enunciated property to be false; and its contrary to be true. He reasons from the assumed truth of this false property, till he arrives at a conclusion dependent upon that assumption, which is contrary to the original hypothesis; and thence it is inferred that the assumption being incompatible in its consequences with the original conditions of the theorem, those conditions and that assumption cannot co-exist. If, then, all the alternatives of the alleged property be thus examined, and thereby excluded from compatibility with the original hypothesis, it will necessarily follow, that this property itself is true. Thus in Euc. I. 25, where one included angle BAC is alleged in the enunciation to be greater than the other EDF, under the hypothesis of BA, AC, being respectively equal to ED, DF, but BC greater than EF: instead of proving the assertion itself, he admits, that in the first place the angle BAC is equal to the angle EDF, and in the second, that it is less. The consequences of these admissions are both shewn to be incompatible with the hypothesis, and hence it is inferred that the angle BAC can neither be equal to EDF, nor less than it. Wherefore as these are the only alternatives to the truth of the enunciation, and both these are false, it follows that the alleged relation of the angles BAC, EDF is true.

This method of proof occurs very frequently in the first and third books of Euclid, and occasionally in the fifth and sixth. In the eleventh and twelfth, it also occurs frequently; and it may be remarked generally, that it occurs more often in the outset of the developement of a system of truths than in the more advanced parts, or in the more recondite theorems.

It must naturally have occurred to Geometers, who were familiar with the use of this mode of assumption, to inquire: "What would be the effect of supposing the alleged theorem to be true, instead of false?" He who first asked this question made the first step in the Geometrical Analysis. He would see at once that the conclusion ought to be consistent with the hypothesis, and with all previously known properties of the hypothetical figure. He may, indeed, find it of little convenience, often of none, in suggesting a direct proof of a very elementary theorem, but as he would be of course led to try its efficacy in more complex cases, he would be gradually impressed with the facts:—that in many cases his steps were merely the reversal of the steps which he had employed in the hypothetic demonstration of the theorem; and that in all cases, a reversal in the order of the steps of his analysis would constitute a synthetic demonstration, though perhaps different from any one previously known to him. He had then discovered the true principle of the *Geometrical Analysis of Theorems;* and it would require but little additional skill to reduce the whole process to a complete system. It is then probable that his discovery would lead to some such rules as the following:—

(1) Assume that the Theorem is true.

(2) Proceed to examine any consequences that result from this admission, by the aid of other truths respecting the figure which have been already proved.

(3) Examine whether any of these consequences be themselves such as are already known to be *true*, or to be *false*.

(4) If any one of them be false, we have arrived at a *reductio ad absurdum*, which proves that the theorem itself is false, as in Euc. I. 25.

(5) If none of the consequences so deduced be *known* to be either true or false, proceed to deduce other consequences from all or any of these, as in (2).

(6) Examine these results, and proceed as in (3) and (4); and if still without any conclusive indications of the truth or falsehood of the alleged theorem, proceed still further, until such are obtained.

In the case of the theorem being false, we shall ultimately arrive at some result contradictory either to the original hypothesis, or to some truth depending upon it. Euclid's indirect demonstrations always end with a contradiction to the immediate hypothesis; but as the propositions to which he applies the method are so extremely elementary, this could scarcely happen otherwise, as, so far, deductions would be made from the hypothesis by direct steps. Where, however, we find a contradiction in our results to any of the consequences of the hypothesis, our conclusion, that the theorem is false, is as legitimate as though the contradiction had immediately been of the hypothesis itself. Nevertheless, if it should be imposed as a rule, that the contradiction shall be of the hypothesis itself, it only requires that we reverse the hypothesis of the property which is so contradicted, employing the contradiction instead of the conclusion of that property; and we shall thus have carried back that contradictory result into direct contrast with the original hypothesis.

It may sometimes happen that our attempts thus to analyse a theorem may be carried on through a considerable number of successive steps, and yet no conclusive evidence of the truth or falsehood of the alleged theorem present themselves. Nor can we ever judge, *à priori*, whether we should succeed by continuing the process further in any one particular direction. Under one aspect this may be considered an inconvenience; but even were it a real inconvenience, it is inevitable, and must so far be taken as a

drawback upon the value of the method. The inconvenience is, however, more apparent than real; or, at least, the inconvenience is amply compensated by the advantages it otherwise confers, not indeed in reference to the demonstration of the proposed theorem, but in its extension of geometrical discovery. A mistake might occur in the synthetic deduction of a proposed theorem, or the theorem might be a mistaken inference from analogy, or from the contemplation of carefully drawn diagrams; but it does not often happen that a theorem is proposed for solution, of the truth of which the proposer has not satisfied himself. The probabilities then are greatly in favour of such proposition being correct. Now in this case, all the investigations which have been made with a view to the *analysis* of that theorem will become so many *synthetic demonstrations* of the results which have been obtained during those unsuccessful attempts to analyse. It will in general be found, too, that they are of such a character as it would scarcely have occurred to any Geometer to adopt with pure reference to synthetic purposes. There can, in fact, be little doubt that the greater part of the most profound and original theorems that are found in the writings of the greatest Geometers of ancient and of modern times, have originated in attempts to analyse some proposed theorem; and which have failed merely from the direction which was pursued, lying in that of the more recondite instead of the more simple order of truths connected with the proposed one. Such failures should therefore be always carefully preserved, till the proposition itself, from which they were deduced, be proved either to be true or false.

Should the course of analysing pursued in the first instance not promise to succeed, by the conclusions becoming more and more elementary in their character, some other properties of the figure connected with the assumed truth should be tried in the same manner; and if this should also fail to accomplish the immediate object, the investigations should be pursued as before.

It has occasionally, though extremely seldom, happened that several such attempts have failed in succession. Yet some mode of deduction *must necessarily* become a true analysis of the theorem; and this will always result from adequate perseverance in these attempts. All the results obtained in the preceding efforts to analyse the theorem, will then constitute a circle of truths, connected with each other by the medium of that one from which they all, as it were, radiate; and often among truths so related, a general principle may be detected, that shall prove of the utmost value in the treatment of entire classes of propositions that now stand in an uninteresting state of isolation from each other. Moreover, by systematizing the propositions of Geometry, we simplify their didactic developement; and by contemplating such attempted analyses of single theorems, if taken in connexion with each other, very great benefits may be conferred upon Geometrical Science and its practical applications.

*There is not the slightest difference between analysis and synthesis, as far as the course of consecutive deduction is concerned.* Both are direct applications of the ordinary enthymeme; and both require the same specific habits of mind, and the same resources as regards truths already known. The only difference consists, as far as mere imagining, reasoning is concerned, in the difference of the starting points of the investigation. In Synthesis we start from the enunciated property as a truth temporarily admitted; and ultimately arrive at some property which we previously knew to be true of the hypothetical figure. We have only to reverse the order of the Syllogisms, and of the subject and predicate in each of them, to convert the analysis into the synthesis in one case, or the synthesis into the analysis in the other. They are so connected, in fact, that had the hypothesis of the proposed theorem been already proved by one process, the analysis of which we have spoken, would have become the synthesis of the other property.

It must now be obvious that the synthesis of the theorem can be at once formed from the analysis, by the reversal of the steps already described, that the analysis may, if

desirable, be altogether suppressed. On the other hand, for all the purposes of giving full and legitimate conviction of the truth of a theorem, the analysis is always sufficient, without adding the synthesis. It is, however, desirable, in a course of Geometrical study, to complete the formal draft of the investigation both in the analytic and synthetic form.

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## THE ANALYSIS OF PROBLEMS.

In every Geometrical operation we perform in the construction of a Problem, we have in mind some precedent reason,—a knowledge of some properties of the figure, either axiomatic or not, which would result from that operation, and a preception of its tendency towards accomplishing the object proposed in the Problem. Our processes for construction are founded on our knowledge of the properties of the figure, *supposed to exist already*, subjected to the conditions which are enunciated in the Proposition itself. No Problem could be constructed (except by mere trials, and verified by mere instrumental experiments) antecedently to the admission of our knowledge of some properties of the figure which it is proposed to construct. The simple reason for the operations employed, is, that they collectively and ultimately fulfil the prescribed conditions; and their so fulfilling the conditions, is only known by previously reasoning upon the figure supposed already to be so constructed as to embody those conditions. Let any Problem be selected from Euclid, and at each step of the operation, let the question be asked, "Why that step is taken?" It will in all cases be found that it is *because* of some known property of the figure required, either in its complete or intermediate states, of which the inventor of the construction must have been in possession. This antecedency of Theorems to all Geometrical construction in Scientific Geometry is universal and essential to its nature.

Let the construction of Euc. IV. 10 be taken in illustration of what has been stated. There are five operations specified in the construction:—

(1) Take *any* line AB.
(2) Divide that line in C, so that, &c.
(3) Describe the circle BDE with centre A and radius AB.
(4) Place BD in that circle, equal to AC.
(5) Join the points A, D.

Why should either of these operations be performed rather than any others? And what clue have we to enable us to foresee that the result of them will be such a triangle as was required? The demonstration affixed to it by Euclid, does undoubtedly prove that these operations must, in conjunction, produce such a triangle: but we are furnished in the Elements with no obvious reason for the adoption of these steps, except we suppose them accidental. To suppose that all the constructions, even the simple ones, were the result of accident only, would be supposing more than could be shewn to be admissible. No construction of the problem could have been devised without a previous knowledge of some of the properties of the figure which was to be produced. In fact, in directing the figure to be constructed, we assume the possibility of its existence; and we study the properties of such a figure on the hypothesis of its actual existence. It is this study of the properties of the figure *that constitutes the Analysis of the problem.*

Let then the existence of a triangle BAD be admitted which has each of the angles ABD, ADB double of the angle BAD, in order to ascertain any properties it may possess which would assist in the actual construction of such a triangle.

Then, since the angle ADB is double of BAD, if we draw a line DC to bisect ADB

and meet AB in C, the angle ADC will be equal to CAD; and hence (Euc. I. 6) the sides AC, CD are equal to one another.

Again, since we have three points A, C, D, not in the same straight line, let us examine the effect of describing a circle through them: that is, describe the circle ACD about the triangle ACD (Euc. IV. 5).

Then, since the angle ADB has been bisected by DC, and since ADB is double of DAB, the angle CDB is equal to the angle DAC in the alternate segment of the circle; the line BD therefore coincides with a tangent to the circle at D (converse of Euc. III. 32).

Whence it follows that the rectangle contained by AB, BC, is equal to the square of BD. (Euc. III. 36.)

But the angle BCD is equal to the two interior opposite angles CAD, CDA; or since these are equal to each other; BCD is the double of CAD, that is of BAD. And since ABD is also double of BAD, by the conditions of the triangle, the angles BCD, CBD are equal, and BD is equal to DC, that is, to AC.

It has been proved that the rectangle AB, BC, is equal to the square of BD; and hence the point C in AB, found by the intersection of the bisecting line DC, is such, that the rectangle AB, BC is equal to the square of AC. (Euc. II. 11.)

Finally, since the triangle ABD is isosceles, having each of the angles ABD, ADB double of the same angle, the sides AB, AD are equal, and hence the points B, D, are in the circumference of the circle described about A with the radius AB. And since the magnitude of the triangle is not specified, the line AB may be of any length whatever.

From this "Analysis of the problem," which obviously is nothing more than an examination of the properties of such a figure supposed to exist already, it will be at once apparent, *why* those steps which are prescribed by Euclid for its construction, were adopted.

The line AB is taken of any length, *because* the problem does not prescribe any specific magnitude to any of the sides of the triangle: the circle BDE is described about A with the distance AB, *because* the triangle is to be isosceles, having AB for one side, and therefore the other extremity of the base is in the circumference of that circle: the line AB is divided in C so that the rectangle AB, BC shall be equal to the square of AC, *because* the base of the triangle must be equal to the segment AC: and the line AD is drawn, *because* it completes the triangle, two of whose sides AB, BD are already drawn.

A careful examination of this process will point out the true character of the method by which the construction of all problems (except perhaps a few simple ones which involve but very few and very obvious steps) have been invented: although the actual analysis itself has been suppressed or concealed, as amongst the ancient Geometers, it appears to have been the general practice.

It will be inferred at once, that the use of the Analysis in reference to the construction of problems, is altogether indispensable in its actual form, where the problem requires several steps for its construction; as it has been shewn to be virtually (though the operations may in certain simple problems be carried on mentally and almost unsuspectedly) essential to the construction of all problems whatever.

Whenever we have reduced the construction to depend upon problems which have been already constructed, our analysis may be terminated; as was the case where, in the preceding example, we arrived at the division of the line AB in C; this problem having been already constructed as the eleventh of the second book.

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# ON THE THEORY OF TRANSVERSALS.

THE theory of Transversals can be applied with great brevity and readiness in the demonstration of inverse propositions, in which it is required to prove, that three, or more than three, convergent lines, under certain conditions, when produced, shall pass through the same point; or, when three, or more than three points, formed by the intersections of lines drawn in certain directions, are situated in the same straight line. It has appeared desirable to give in detail a few of the principal properties: at the same time, it may be remarked, that the theory of Transversals appears to be not an unsuitable transition from the Ancient to the Modern Geometry, as it involves only Geometrical and Algebraical considerations, without reference to Trigonometry.

A rectilinear transversal is defined to be a straight line which intersects a system of other straight lines, (figures, Prop. I.)

Thus, if a straight line $cba$ be drawn intersecting two sides AB, AC of a triangle ABC, in the points $c$, $b$, and meeting the third side BC produced in $a$; or intersecting the three sides produced in $a$, $b$, $c$; the line $cba$ so drawn is called a transversal.

The transversal divides each of the sides of the triangle, or the sides produced, into two segments which lie between the vertices of the triangle and the transversal.

Thus A$c$, $c$B are the segments of the side AB between the two vertices A, B and the transversal $cba$: A$b$, $b$C the segments of AC; and B$a$, $a$C the segments of BC. Also, A$b$, C$a$, B$c$ and $b$C, $a$B, $c$A are respectively the alternate segments of the sides made by the transversal $cba$.

Any figure formed by the meeting of four straight lines at their extremities, is called a *simple quadrilateral:* thus each of the three figures formed by the four lines B$c$, $cb$, $b$C, CB; AC, C$a$, $ac$, $c$A; AB, B$a$, $ab$, $b$A is called a *simple quadrilateral.*

The *complete figure* formed by the production of the opposite sides of the simple quadrilateral to meet each other, or two adjacent sides to meet the other two sides is called a *complete quadrilateral.*

Thus the *same complete quadrilateral* is formed from each of the three simple quadrilaterals;

(1) By the production of the opposite sides B$c$, C$b$, and BC, $cb$, of the simple quadrilateral B$cb$CB to meet in the points A, $a$.

(2) By the production of the two opposite sides A$c$, $a$C of the simple quadrilateral A$ca$CA to meet in the point B.

(3) By the production of the two adjacent sides A$b$, $ab$ of the simple quadrilateral AB$ab$A to meet the other two adjacent sides in the points C, $c$.

The six points at which every two of the four lines meet or intersect are called the vertices of the complete quadrilateral, and the points A, B, C, $a$, $b$, $c$ are the six vertices of the complete quadrilateral AB$ab$A.

The three straight lines which join every two opposite vertices of a complete quadrilateral are called its diagonals, thus the lines A$a$, B$b$, C$c$ are the three diagonals of the complete quadrilateral AB$ab$A.

The line $cba$ was considered as a transversal intersecting the sides or the sides produced of the triangle ABC.

In a similar manner $AbC$ may be considered as a transversal intersecting the sides of the triangle $cBa$; and the alternate segments of the sides are BA, $cb$, $aC$ and $Ac$, $ba$, CB respectively.

Also $AcB$ is a transversal to the triangle $bCa$; and $aB$, CA, $bc$ and BC, $Ab$, $ca$ are respectively the alternate segments of the sides of the triangle.

And lastly, $aCB$ is a transversal to the triangle $Abc$; and AB, $ca$, $bC$ and $Bc$, $ab$, CA are respectively the alternate segments of the sides.

If a straight line be drawn as a transversal intersecting the four sides of a complete quadrilateral and two of its diagonals; the three pairs of points in which it intersects the two diagonals, and the alternate sides, produced if necessary, of the complete quadrilateral, are called *conjugate points of the transversal*, thus $m$, $m'$; $n$, $n'$; $p$, $p'$; are the three pairs of conjugate points of the transversal $nm$, (figure, Prop. IV.)

## PROPOSITION I.

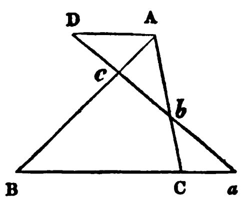
*If a straight line intersect two sides* AB, AC, *of a triangle* ABC *in the points* c, b, *and the base* BC *produced in* a: *or intersect the three sides* AB, AC, BC *produced in the points* c, b, a: *prove that* Ab.Bc.Ca=aB.bC.cA.

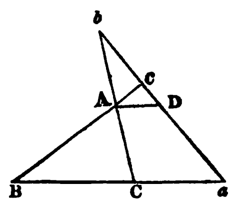
*Conversely. If two points* c, b *be taken in the sides* AB, AC *of a triangle* ABC, *and a third point* a *in the remaining side produced; or if the three points* c, b, a, *be in the three sides produced of the triangle, such that*

$$Ab.Bc.Ca=aB.bC.cA:$$

*the points*, a, b, c *shall be in the same straight line.* (Geom. Ex. Theo. 161. p. 364.)

First. Through A the vertex of the triangle ABC, draw AD parallel to BC.




Then $\dfrac{Bc}{Ac}=\dfrac{aB}{AD}$, by the similar triangles AD$c$, $cBa$;

and $\dfrac{Ab}{Cb}=\dfrac{AD}{Ca}$, by the similar triangles DA$b$, $bCa$;

$$\text{whence } \frac{Ab.Bc}{cA.bC}=\frac{aB}{Ca}.$$

And, therefore, $Ab.Bc.Ca=aB.bC.cA$;

or, the rectangular parallelopipeds contained by the alternate segments intercepted between the vertices of a triangle and a transversal, are equal to one another.

Secondly. Join $cb$, $ba$: if $ba$ be not in the same straight line as $bc$, let $ba'$* be in the same straight line with it.

Then, since $cba'$ is a transversal to the triangle ABC,

$$\text{therefore, } \mathrm{A}b.\mathrm{B}c.\mathrm{C}a' = a'\mathrm{B}.b\mathrm{C}.c\mathrm{A};$$

$$\text{but } \mathrm{A}b.\mathrm{B}c.\mathrm{C}a = a\mathrm{B}.b\mathrm{C}.c\mathrm{A};$$

$$\text{hence } \frac{\mathrm{C}a}{\mathrm{C}a'} = \frac{a\mathrm{B}}{a'\mathrm{B}}; \text{ and } \frac{\mathrm{C}a}{a\mathrm{B}} = \frac{\mathrm{C}a'}{a'\mathrm{B}};$$

$$\text{therefore } \frac{\mathrm{CB}}{a\mathrm{B}} = \frac{\mathrm{CB}}{a'\mathrm{B}}; \text{ and } a\mathrm{B} = a'\mathrm{B};$$

or the point $a'$ coincides with the point $a$; and therefore the three points $c$, $b$, $a$ are in the same straight line.

Cor. 1. The expression $\mathrm{A}b.\mathrm{B}c.\mathrm{C}a = a\mathrm{B}.b\mathrm{C}.c\mathrm{A}$, may be put under the following form, which perhaps will be found the most convenient in practice, as it connects together the ratio of the segments of each side,

$$\frac{\mathrm{A}b}{b\mathrm{C}}\cdot\frac{\mathrm{C}a}{a\mathrm{B}}\cdot\frac{\mathrm{B}c}{c\mathrm{A}} = 1;$$

and this form of the expression may be interpreted to mean, in the language of pure Geometry, that the ratio compounded of the ratios of the segments of the three sides of the triangle taken in order, is a ratio of equality.

Or, in the language of Algebraical ratio; that the product of the ratios of the segments of each of the three sides taken in order, is equal to unity.

If, however, this form be objected to, as not being strictly Geometrical; as well as the preceding one, as involving the conception of a solid, and which ought not to be admitted into considerations on plane Geometry: the expression itself may be retained in the form in which it was deduced from the two proportions,

$$\text{thus, } \frac{\mathrm{A}b.\mathrm{B}c}{c\mathrm{A}.b\mathrm{C}} = \frac{a\mathrm{B}}{\mathrm{C}a},$$

which may be exhibited under the three following forms:

$$a\mathrm{B} : \mathrm{C}a :: \mathrm{A}b.\mathrm{B}c : c\mathrm{A}.b\mathrm{C};$$

$$\text{or } b\mathrm{A} : \mathrm{C}b :: \mathrm{B}a.\mathrm{A}c : a\mathrm{C}.c\mathrm{B};$$

$$\text{or } c\mathrm{B} : \mathrm{A}c :: \mathrm{B}a.\mathrm{C}b : a\mathrm{C}.b\mathrm{A};$$

and expressed in the following terms:

If a transversal be drawn to any triangle, the segments of any one side between the transversal and two vertices of the triangle, are to each other as the ratio compounded of the ratios of the alternate segments of the other two sides: or, as the rectangles contained by the alternate segments of the other two sides.

Cor. 2. If, in the same way as the line $acb$ was considered as a transversal to the triangle ABC, the lines AC, AB, BC be considered as transversals to the triangles $c\mathrm{B}a$, $b\mathrm{C}a$, $\mathrm{A}cb$, respectively; the four following results are obtained:

$$\frac{\mathrm{A}b}{b\mathrm{C}}\cdot\frac{\mathrm{C}a}{a\mathrm{B}}\cdot\frac{\mathrm{B}c}{c\mathrm{A}} = 1; \text{ (I.)} \qquad \frac{\mathrm{BC}}{\mathrm{C}a}\cdot\frac{ab}{bc}\cdot\frac{c\mathrm{A}}{\mathrm{AB}} = 1; \text{ (II.)}$$

$$\frac{a\mathrm{B}}{\mathrm{BC}}\cdot\frac{\mathrm{CA}}{\mathrm{A}b}\cdot\frac{bc}{ca} = 1; \text{ (III.)} \qquad \frac{\mathrm{AB}}{\mathrm{B}c}\cdot\frac{ca}{ab}\cdot\frac{b\mathrm{C}}{\mathrm{CA}} = 1. \text{ (IV.)}$$

These four results are not independent of each other, but any one of the four may be deduced from the remaining three.

* $ba'$ is not drawn in the diagram.

COR. 3. Again, if every two of the three independent results be combined, three other expressions, each consisting of eight segments, are obtained, which will express the relations which subsist between the four lines that form the complete quadrilateral, and their eight segments.

Thus, from I. and II., is deduced $\dfrac{CB.Bc}{cb.bC} = \dfrac{AB.Ba}{Ab.ba}$;

from I. and IV., ......... $\dfrac{BA.Ab}{Ba.ab} = \dfrac{CA.Ac}{Ca.ac}$;

from I. and III., ......... $\dfrac{Bc.cb}{BC.Cb} = \dfrac{Ac.ca}{AC.Ca}$.

COR. 4. From the general expression (I) of the product of the ratios of the segments, let the consequences be deduced; first, when the transversal is parallel to any one of the sides of the triangle: and secondly, when the transversal passes through the vertex A of the triangle, and meets the base or the base produced in the point $a$.

COR. 5. If B C $a$ be considered as a transversal to the three straight lines AB, AC, A$a$, which are drawn through the same point A; then drawing AD perpendicular to $a$B, the following property may be proved:

$$BC.Ca^2 + Ca.BA^2 = AC^2.Ba + BC.Ca.aB.$$

(See Geom. Ex. Theo. 41, p. 355).

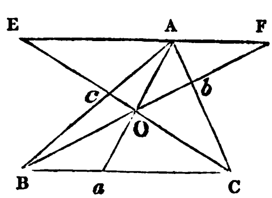
COR. 6. Ascertain whether this theorem holds good when the point A is supposed to fall any where in the transversal B C $a$.

## PROPOSITION II.

*If three straight lines be drawn from the angles of a triangle through any point* O *within the triangle, and be produced to meet the opposite sides in* a, b, c: *prove that* $Ac.Ba.Cb = cB.aC.bA$.

*Conversely. If three straight lines be drawn from the angles of a triangle* ABC *to meet the opposite sides in the points* a, b, c, *so that* $Ac.Ba.Cb = cB.aC.bA$: *then the three straight lines so drawn shall pass through the same point.* (Geom. Ex. Theo. 160, p. 364.)

First. Through the vertex A draw EAF parallel to BC, and meeting B$b$, C$c$ produced, in F, E respectively.




Then $\dfrac{AE}{aC} = \dfrac{AO}{Oa}$, by the similar triangles EAO, O$a$C;

$= \dfrac{AF}{Ba}$, by the similar triangles AFO, O$a$B;

Hence $\frac{Ba}{aC} = \frac{AF}{AE}$;

Again, $\frac{Cb}{bA} = \frac{BC}{AF}$, by the similar triangles $AbF$, $CbB$;

also, $\frac{Ac}{cB} = \frac{AE}{BC}$, by the similar triangles $AcE$, $BcC$;

$$\text{whence } \frac{Ac}{cB} \cdot \frac{Ba}{aC} \cdot \frac{Cb}{bA} = 1;$$

$$\text{or } Ac.Ba.Cb = cB.aC.bA.$$

Secondly. Let $Aa$, $Bb$ intersect each other in the point O; and let $c$ be such a point in AB, that $Ac.Ba.Cb = cB.aC.bA$.

Join CO, then CO produced, passes through the point $c$.

If CO produced do not pass through the point $c$, let it pass through $c'$ *, some other point in AB.

Since O is a point within the triangle, and $Aa$, $Bb$, $Cc'$, are drawn through it, and meet the sides of the triangle in $a$, $b$, $c'$; therefore

$$Ac'.Ba.Cb = c'B.aC.bA.$$

But $Ac.Ba.Cb = cB.aC.bA$, by hypothesis;

$$\text{therefore } \frac{Ac'}{Ac} = \frac{c'B}{cB}, \text{ and } \frac{Ac'}{c'B} = \frac{Ac}{cB};$$

$$\text{whence } \frac{AB}{c'B} = \frac{AB}{cB}, \text{ and } c'B = cB;$$

or, the point $c'$ coincides with the point $c$, and therefore $COc'$ coincides with $COc$, and the three lines $Aa$, $Bb$, $Cc$, pass through the same point O.

This Proposition is also true when the point O is outside the triangle and on either side of the base BC; also the expression $Ac.Ba.Cb = cB.aC.bA$ may be put under the forms:

$$Ac : cB :: aC.bA : Ba.Cb;$$
$$Ca : aB :: cA.bC : Bc.Ab;$$
$$Cb : bA :: cB.aC : Ac.Ba;$$

and expressed in the following terms:

If three straight lines be drawn from the angles of a triangle through any point to meet the opposite sides, or the opposite sides produced; the segments of each side are in the same ratio, as the ratio compounded of the ratios of the alternate segments of the other two sides.

COR. 1. The following relations may also be shewn to exist between the six lines and their twelve segments.

(1) When $AOa$ is taken as a transversal to the triangles $BCc$, $CBb$,

$$\frac{Ba}{aC} \cdot \frac{CA}{Ab} \cdot \frac{bO}{OB} = 1; \qquad \frac{Ba}{aC} \cdot \frac{CO}{Oc} \cdot \frac{cA}{AB} = 1.$$

(2) When $BOb$ is taken as a transversal to the triangles $ACc$, $CAa$,

$$\frac{CO}{Oc} \cdot \frac{cB}{BA} \cdot \frac{Ab}{bC} = 1; \qquad \frac{Cb}{bA} \cdot \frac{AO}{Oa} \cdot \frac{aB}{BC} = 1.$$

* $Oc'$ is not drawn in the diagram.

(3) When $COc$ is taken as a transversal to the triangles $ABb$, $BAa$,

$$\frac{Bc}{cA}\cdot\frac{AC}{Cb}\cdot\frac{bO}{OB}=1;\qquad \frac{Ac}{cB}\cdot\frac{BC}{Ca}\cdot\frac{aO}{OA}=1.$$

(4) When $AcB$ is taken as a transversal to the triangles $COa$, $COb$,

$$\frac{Cc}{CO}\cdot\frac{OA}{Aa}\cdot\frac{aB}{BC}=1;\qquad \frac{Cc}{cO}\cdot\frac{OB}{Bb}\cdot\frac{bA}{AC}=1.$$

(5) When $AbC$ is taken as a transversal to the triangles $BOa$, $BOc$,

$$\frac{Bb}{bO}\cdot\frac{OA}{Aa}\cdot\frac{aC}{CB}=1;\qquad \frac{Bb}{bO}\cdot\frac{OC}{Cc}\cdot\frac{CA}{AB}=1.$$

(6) When $BaC$ is taken as a transversal to the triangles $AOb$, $AOc$,

$$\frac{Aa}{aO}\cdot\frac{OB}{Bb}\cdot\frac{bC}{CA}=1;\qquad \frac{Aa}{aO}\cdot\frac{OC}{Cc}\cdot\frac{cB}{BA}=1.$$

These twelve relations are deduced when the point O is considered to be within the triangle; they are also true when the point O is outside the triangle and on either side of the base.

It is also worth while to ascertain how many of these properties form independent conditions of relation between the lines and their segments.

Cor. 2. In a similar manner, if the three diagonals of any complete quadrilateral be drawn, it will be found that the figure contains thirty-three simple quadrilaterals, and forty-four relations may be deduced from them by means of Prop. I. and II.

Cor. 3. By combining the relation proved in Prop. II. with the second, third, and sixth in Cor. 1: the following relation between the sides of the triangle, the transversals, and their segments is deduced

$$\frac{AO}{Oa}\cdot\frac{BO}{Ob}\cdot\frac{CO}{Oc}=\frac{AB}{Ab}\cdot\frac{BC}{Bc}\cdot\frac{CA}{Ca}.$$

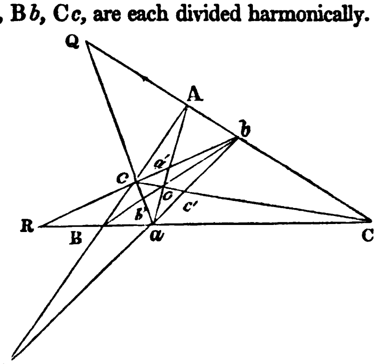
## PROPOSITION III.

*If through a given point within a triangle lines are drawn from the angles to the opposite sides, and the points of section be joined, the first three lines will be harmonically divided.* (Geom. Ex. Theo. 72, p. 357.)

Let $Aa$, $Bb$, $Cc$ be drawn through any point O within the triangle ABC meeting the opposite sides in $a$, $b$, $c$.

Draw $ab$, $bc$, $ca$, intersecting $Cc$, $Aa$, $Bb$, in $c'$, $a'$, $b'$, respectively.

Then the lines $Aa$, $Bb$, $Cc$, are each divided harmonically.



From the point $b$, three straight lines $b\mathrm{A}$, $bc$, $b\mathrm{O}$, are drawn to the angles of the triangle $\mathrm{A}c\mathrm{O}$:

$$\text{therefore } \frac{\mathrm{AB}}{\mathrm{B}c}\cdot\frac{c\mathrm{C}}{\mathrm{CO}}\cdot\frac{\mathrm{O}a'}{a'\mathrm{A}}=1, \text{ by Prop. II.,}$$

$$\text{and } \frac{\mathrm{AB}}{\mathrm{B}c}\cdot\frac{c\mathrm{C}}{\mathrm{CO}}\cdot\frac{\mathrm{O}a}{a\mathrm{A}}=1,$$

for $\mathrm{B}a\mathrm{C}$ is a transversal to the triangle $\mathrm{A}c\mathrm{O}$;

$$\text{Hence } \frac{\mathrm{O}a}{a\mathrm{A}}=\frac{\mathrm{O}a'}{a'\mathrm{A}},$$

$$\text{and } \frac{\mathrm{A}a}{\mathrm{O}a}=\frac{\mathrm{A}a'}{\mathrm{O}a'}=\frac{\mathrm{A}a-aa'}{aa'-\mathrm{O}a},$$

$$\text{or } \mathrm{A}a : \mathrm{O}a :: \mathrm{A}a-aa' : aa'-\mathrm{O}a;$$

wherefore $\mathrm{A}a$, $aa'$, $\mathrm{O}a$ are in harmonical proportion; or the line $\mathrm{A}a$ is divided harmonically in the points $a'$, O.

In a similar way, it may be shewn that $\mathrm{B}b$, $bb'$, $\mathrm{O}b'$, are in harmonical proportion; as also $\mathrm{C}c$, $cc'$, $\mathrm{O}c$.

Cor. 1. If the lines $ab$, $bc$, $ca$ be produced to meet the three sides of the triangle produced in the points P, Q, R; the lines $b\mathrm{P}$, $a\mathrm{Q}$, $b\mathrm{R}$, are divided harmonically in the points $c'$, $a$; $b'$, $c$; $a'$, $c$; respectively: as also AP, CQ, CR, the sides produced of the triangle, in the points, $c$, B; $b$, A; $a$, B; respectively.

First. From the point A, the straight lines $\mathrm{A}b$, AO, $\mathrm{A}c$ are drawn to $b$, O, $c$, the angular points of the triangle $bc\mathrm{O}$, and these lines cut the side $bc$ in $a'$ and meet the other sides $b\mathrm{O}$, $c\mathrm{O}$ produced in B, C;

$$\text{therefore } \frac{ba'}{a'c}\cdot\frac{c\mathrm{C}}{\mathrm{CO}}\cdot\frac{\mathrm{OB}}{\mathrm{B}b}=1, \text{ by Prop. II.}$$

And the transversal RC intersects the sides produced of the same triangle $bc\mathrm{O}$;

$$\text{therefore } \frac{b\mathrm{R}}{\mathrm{R}c}\cdot\frac{c\mathrm{C}}{\mathrm{CO}}\cdot\frac{\mathrm{OB}}{\mathrm{B}b}=1, \text{ by Prop. I.}$$

$$\text{Hence } \frac{b\mathrm{R}}{\mathrm{R}c}=\frac{ba'}{a'\mathrm{C}}=\frac{b\mathrm{R}-\mathrm{R}a'}{\mathrm{R}a'-\mathrm{R}c};$$

or $b\mathrm{R}$, $\mathrm{R}a'$, $\mathrm{R}c$ are in harmonical proportion.

Similarly, it may be shewn that $\mathrm{P}b$ is divided harmonically in $a$, $c'$; as also $\mathrm{Q}a$ in $c$, $b'$.

$$\text{Secondly. Since } \frac{\mathrm{B}c}{c\mathrm{A}}\cdot\frac{\mathrm{A}b}{b\mathrm{C}}\cdot\frac{\mathrm{C}a}{a\mathrm{B}}=1, \text{ by Prop. II.,}$$

$$\text{and } \frac{\mathrm{A}b}{b\mathrm{C}}\cdot\frac{\mathrm{C}a}{a\mathrm{B}}\cdot\frac{\mathrm{BP}}{\mathrm{PA}}=1, \text{ by Prop. I.,}$$

For the transversal $\mathrm{P}ab$ intersects the triangle ABC.

$$\text{Hence } \frac{\mathrm{BP}}{\mathrm{PA}}=\frac{\mathrm{B}c}{c\mathrm{A}},$$

$$\text{and } \frac{\mathrm{AP}}{\mathrm{PB}}=\frac{c\mathrm{A}}{\mathrm{B}c}=\frac{\mathrm{PA}-\mathrm{P}c}{\mathrm{P}c-\mathrm{PB}};$$

or AP, $\mathrm{P}c$, PB are in harmonical proportion, and the line PA is divided harmonically in $c$, B.

In a similar way, it may be shewn, that CQ, and CR, are harmonically divided in $b$, A; $a$, B, respectively.

Cor. 2. Since the lines $\mathrm{A}a'\mathrm{O}a$, $ba'c\mathrm{R}$, $\mathrm{C}a\mathrm{BR}$; $\mathrm{B}b'\mathrm{O}b$, $ab'c\mathrm{Q}$, $\mathrm{C}b\mathrm{AQ}$; $\mathrm{C}c'\mathrm{O}c$, $bc'a\mathrm{P}$, $\mathrm{A}c\mathrm{BP}$; are the diagonals produced of the three complete quadrilaterals BACOB, ABCOA, ACBOA, respectively: the results of Prop. III. and Cor. 1. may be generally expressed in the following terms:

If the three diagonals of a complete quadrilateral be drawn and be produced to meet one another; each of the diagonals is divided harmonically by the other two.

Cor. 3. The three points P, Q, R, are in the same straight line.

For considering $\mathrm{P}ab$, $\mathrm{Q}ca$, $\mathrm{R}cb$, as transversals to the triangle ABC, by Prop. I. we obtain;

$$\frac{\mathrm{AP}}{\mathrm{PB}}\cdot\frac{\mathrm{B}a}{a\mathrm{C}}\cdot\frac{\mathrm{C}b}{b\mathrm{A}}=1, \text{ or } \frac{\mathrm{AP}}{\mathrm{PB}}=\frac{\mathrm{A}b}{b\mathrm{C}}\cdot\frac{\mathrm{C}a}{a\mathrm{B}};$$

$$\frac{\mathrm{CQ}}{\mathrm{QA}}\cdot\frac{\mathrm{A}c}{c\mathrm{B}}\cdot\frac{\mathrm{B}a}{a\mathrm{C}}=1, \text{ or } \frac{\mathrm{CQ}}{\mathrm{QA}}=\frac{\mathrm{C}a}{a\mathrm{B}}\cdot\frac{\mathrm{B}c}{c\mathrm{A}};$$

$$\frac{\mathrm{BR}}{\mathrm{RC}}\cdot\frac{\mathrm{C}b}{b\mathrm{A}}\cdot\frac{\mathrm{A}c}{c\mathrm{B}}=1, \text{ or } \frac{\mathrm{BR}}{\mathrm{RC}}=\frac{\mathrm{B}c}{c\mathrm{A}}\cdot\frac{\mathrm{A}b}{b\mathrm{C}}.$$

$$\text{Hence } \frac{\mathrm{AP}}{\mathrm{PB}}\cdot\frac{\mathrm{BR}}{\mathrm{RC}}\cdot\frac{\mathrm{CQ}}{\mathrm{QA}}=\left(\frac{\mathrm{C}a}{a\mathrm{B}}\cdot\frac{\mathrm{B}c}{c\mathrm{A}}\cdot\frac{\mathrm{A}b}{b\mathrm{C}}\right)^2.$$

$$\text{But } \frac{\mathrm{C}a}{a\mathrm{B}}\cdot\frac{\mathrm{B}c}{c\mathrm{A}}\cdot\frac{\mathrm{A}b}{b\mathrm{C}}=1, \text{ by Prop. II.};$$

$$\text{therefore } \frac{\mathrm{AP}}{\mathrm{PB}}\cdot\frac{\mathrm{BR}}{\mathrm{RC}}\cdot\frac{\mathrm{CQ}}{\mathrm{QA}}=1;$$

which is the condition fulfilled when a straight line is drawn intersecting the three sides of a triangle, AB, CA, CB produced, in the points P, Q, R.

Hence the three points P, Q, R, are in the same straight line.

Cor. 4. The lines $\mathrm{A}a$, $\mathrm{B}b$, $\mathrm{C}c$ may be considered as transversals to the triangle $abc$, and a similar series of relations may be deduced, as in Prop. II.

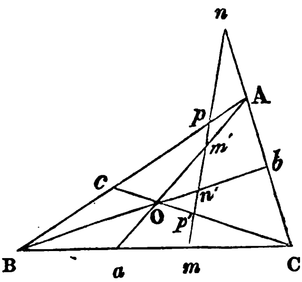
## PROPOSITION IV.

*If three lines be drawn from the angles* A, B, C *of a triangle through any point* O *to meet the opposite sides in the points* a, b, c; *and if a transversal be drawn intersecting these lines in* m′, n′, p′, *and the sides of the triangle, produced if necessary, in the points* m, n, p: *then the three following relations exist between the parts into which the transversal is divided.*

$$\frac{mp}{pm'}\cdot\frac{m'n}{nm}\cdot\frac{mp'}{p'm'}\cdot\frac{m'n'}{n'm}=1; \quad (1) \qquad \frac{nm'}{mn'}\cdot\frac{n'p}{pn}\cdot\frac{nm}{mn'}\cdot\frac{n'p'}{p'n}=1; \quad (2)$$

$$\frac{mp}{pn'}\cdot\frac{n'p'}{p'm'}\cdot\frac{m'p}{pn}\cdot\frac{np'}{p'm}=1; \quad (3).$$

First, the triangle $amm'$ is intersected by the transversals AB, B$b$,




$$\text{therefore } \frac{m'A}{Aa}\cdot\frac{aB}{Bm}\cdot\frac{mp}{pm'}=1; \qquad \frac{mB}{Ba}\cdot\frac{aO}{Om'}\cdot\frac{m'n'}{n'm}=1.$$

Again, the same triangle is intersected by the transversals AC, C$c$,

$$\text{therefore } \frac{m'n}{nm}\cdot\frac{mC}{Ca}\cdot\frac{aA}{Am'}=1; \qquad \frac{m'O}{Oa}\cdot\frac{aC}{Cm}\cdot\frac{mp'}{p'm'}=1,$$

$$\text{whence is deduced } \frac{mp}{pm'}\cdot\frac{m'n}{nm}\cdot\frac{mp'}{p'm'}\cdot\frac{m'n'}{n'm}=1.$$

Secondly. In a similar way since the triangle $nn'b$ is intersected by the transversals BA, A$a$; and by BC, C$c$; four relations arise, from which may be deduced

$$\frac{nm'}{m'n'}\cdot\frac{n'p}{pn}\cdot\frac{nm}{mn'}\cdot\frac{n'p'}{p'n}=1.$$

Thirdly. The triangle $pp'c$ is intersected by the transversal CA, A$a$; and by CB, B$b$; other four relations are found from which there results

$$\frac{mp}{pn'}\cdot\frac{n'p'}{p'm'}\cdot\frac{m'p}{pn}\cdot\frac{np'}{p'm}=1.$$

Cor. From these three relations, each involving eight of the segments of the transversal, a relation may be found which involves only six segments; multiplying these results together, is deduced

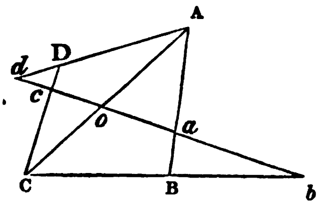
$$\frac{mp'}{p'n}\cdot\frac{nm'}{m'p}\cdot\frac{pn'}{n'm}=1.$$

Three other forms may be deduced from the same three expressions, (I.) by multiplying (1) and (2) together and dividing by (3); (II.) by multiplying (1) and (3) together and dividing by (2); (III.) by multiplying (2) and (3) together and dividing by (1).

## PROPOSITION V.

*If any polygon be intersected by a transversal, the segments of the sides have to each other a relation similar to that of the segments of the sides of a triangle.*

Let ABCD be a polygon of four sides, and let its opposite sides AB, CD be intersected in $b$, $c$ by a transversal which meets AD, CB produced in $d$, $b$.




Join AC intersecting the transversal $db$ in O.

Then the transversal $dc$O intersects the triangle ACD,

$$\text{therefore } \frac{Ad}{dD} \cdot \frac{Dc}{cC} \cdot \frac{CO}{OA} = 1, \quad \text{Prop. I.}$$

And the transversal $ba$O intersects the triangle ABC,

$$\text{therefore } \frac{Aa}{aB} \cdot \frac{Bb}{bC} \cdot \frac{CO}{OA} = 1. \quad \text{Prop. I.}$$

$$\text{Whence } \frac{Aa}{aB} \cdot \frac{Bb}{bC} \cdot \frac{Cc}{cD} \cdot \frac{Dd}{dA} = 1,$$

or, the ratio compounded of the ratios of the segments of the four sides of the polygon taken in order, is a ratio of equality.

This result may also be expressed thus

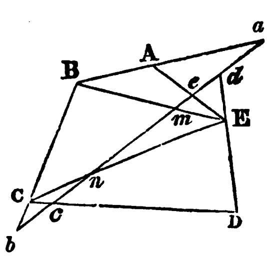
$$Aa.Bb.Cc.Dd = aB.bC.cD.dA,$$

or it may be exhibited in the form of a proportion, for instance

$$\frac{Aa.Cc}{aB.cD} = \frac{bC.dA}{bB.dD}.$$

The transversal $b$O$acd$ was supposed in the preceding result to intersect AB, CD, *two opposite* sides of the polygon; the same property is also true when the transversal intersects *two adjacent* sides, or the *four sides produced.*

Let ABCDE be a polygon of five sides, and let AE, CD, any two sides of it, be intersected by a transversal in the points $e$, $c$: let BA, BC, DE produced, meet the transversal in $a$, $b$, $d$, respectively. Draw the diagonals EB, EC intersecting the transversal in $m$, $n$,




then $\frac{Em}{mB} \cdot \frac{Ba}{aA} \cdot \frac{Ae}{eE} = 1$, for $mea$ intersects the triangle ABE,

$$\frac{Bm}{mE}\cdot\frac{En}{nC}\cdot\frac{Cb}{bB}=1,\ ncb \text{ intersects BCE,}$$

$$\frac{Cn}{nE}\cdot\frac{Ed}{dD}\cdot\frac{Dc}{cC}=1,\ cnd \text{ intersects CDE.}$$

$$\text{Whence } \frac{Ba}{aA}\cdot\frac{Ae}{eE}\cdot\frac{Ed}{dD}\cdot\frac{Dc}{cC}\cdot\frac{Cb}{bB}=1;$$

or, as before, the ratio compounded of the ratios of the segments of the sides of the figure successively taken in order is a ratio of equality.

In a similar manner, for any polygon ABCDEF of six sides, when intersected by a transversal, adopting the same notation as for the polygon of five sides, it may be proved, that

$$\frac{Ba}{aA}\cdot\frac{Af}{fF}\cdot\frac{Fe}{eE}\cdot\frac{Ed}{dD}\cdot\frac{Dc}{cC}\cdot\frac{Cb}{bB}=1;$$

or the ratio compounded of the ratios of the segments of the six sides taken in order is a ratio of equality.

And similarly for any polygon whatever, whether the transversal intersect two of the sides of the polygon or only the sides produced.

Cor. If the hexagon be inscriptible in a circle, then the three points of intersections of the alternate sides produced, are in the same straight line.

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Carnot was the first who systematically pointed out the importance of the *relations of position* in Geometrical figures. This he exhibited in a tract entitled "*De la Correlation des figures de Géométrie,*" published in 1801. His greater work on the subject appeared in 1803, under the title of "*Géométrie de Position,*" and three years afterwards, in 1806, his tract "*Sur la Théorie des Transversales.*" The subject has been both extended and simplified by subsequent writers; among whom may be named C. J. Brianchon, who published, in 1818, his "*Application de la Théorie des Transversales.*" Many interesting properties of Transversals will be found in the notes of M. Chasles's "*Aperçu Historique des Methodes en Géométrie.*" The only English work in which the subject is systematically treated, is the last edition of Hutton's Course of Mathematics, in which the theory is applied both to Elementary Geometry, and to properties of the Conic Sections.

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## HINTS, &c. TO THE PROBLEMS. BOOK I.

2. There is another method whereby a line may be divided into three equal parts:—by drawing from one extremity of the given line, another making any acute angle with it, and taking three equal distances from the extremity, then joining the extremities, and through the other two points of division, drawing lines parallel to this line through the other two points of division, and to the given line; the three triangles thus formed are equal in all respects. This may be extended for any number of parts.

5. This is a particular case of Euc. I. 22.

6. The same remark applies.

7. Let A be the given point without the line, and B the given point in the line. Suppose D the point required in the line. If AB and AD be joined, DAB is an isosceles triangle: draw DE perpendicular to AB, and the construction is obvious.

8. This may be effected in two ways, (1) by Euc. I. 9, 10, when the two lines meet; (2) when the lines are produced beyond the point of intersection, by Euc. I. 9, 31. It may be remarked also, that the line drawn through the given point makes equal angles with the two given lines, Euc. I. 5.

9. Suppose the two lines CB, DB to meet in B, and that EAF drawn through the given point A, is bisected in that point. Through A, draw AG parallel to BC, and through G, GH parallel to EF. Then EAGH is a parallelogram, and BG is equal to GF. Hence the synthesis.

10. Apply Euc. I. 1, 9.

11. The angle to be trisected is one-fourth of a right angle. If an equilateral triangle be described on one of the sides of the triangle which contains the given angle, and a line be drawn to bisect that angle of the equilateral triangle which is at the given angle, the angle contained between this line and the other side of the triangle, will be one twelfth of a right angle, or equal to one third of the given angle.

It may be remarked, generally, that any angle which is the half, fourth, eighth, &c. part of a right angle, may be trisected by Plane Geometry.

12. This may be done by means of Euc. I. 47, taking three lines of 3, 4, 5 units. Or, by means of *equal* straight lines, figure Euc. I. 1, if an equilateral triangle CBF be described on CB, and another CFG on CF, the line drawn from G to B is at right angles to AB. Or, take a point D in AB, draw DE equal to DC and inclined to AB, C being the given point. Join EC, and produce it till CG be equal to CD; make CF equal to CE, join FG and produce it till GH be equal to GC. Then CH will be perpendicular to AB.

13. Take any point in the given line, and apply Euc. I. 23, 31.

14. The only distinction between this and Prob. 8, above, is in the quæsitum. The quæsita in both are *essentially* connected.

15. The two given points may be both on the same side, or one point may be on each side of the line. If the point required in the line be supposed to be found, and lines be drawn joining this point and the given points, an isosceles triangle is formed, and if a perpendicular be drawn on the base from the point in the line: the construction is obvious.

16. The line to which the given line is to be parallel, may pass through an angle of the triangle, or it may not; if it do not, draw through an angle of the triangle a line parallel to this line, Euc. I. 31. If a line equal to the given line be supposed to be drawn parallel to the line through one of the angles, a parallelogram may be formed and the construction determined.

17. The construction of this problem may be effected from Prob. 1, p. 293. ?

18. If the point in the base be supposed to be determined, and lines drawn from it parallel to the sides, it will be found to be in the line which bisects the vertical angle of the triangle.

19. This may be effected by Prob. 3, p. 295.

20. The line required is obviously the diagonal joining the two obtuse angles.

21. If AC be drawn from A one extremity of the given line AB making *any*

angle less than the given angle, and if from A, a line AD be drawn making with AC an angle equal to the given angle, and BC, BD be drawn from B parallel to AD, AC respectively:—the parallelogram is formed with the given conditions. The Problem is indeterminate.

22. If in the figure Euc. I. 1, the circles intersect each other in C, C′, and CA, AC′, C′B, BC be joined, ACBC′ is the parallelogram required.

23. See the figure, Euc. I. 35.

24. Construct a right-angled parallelogram by Euc. I. 44, equal to the given quadrilateral figure, and from one of the angles draw a line to meet the opposite side and equal to the base of the rectangle, and a line from the adjacent angle parallel to this line will complete the rhombus.

25. Bisect BC in D, and through the vertex A draw AE parallel to BC, with centre D and radius equal to half the sum of AB, AC, describe a circle cutting AE in E.

26. Make a triangle ABC equal to the given figure, by Prob. 27. Produce the base BC, if necessary, making BD equal to the given base. On BD make the triangle BDE equal to ABC; through E draw EG parallel to BD, and upon BD describe a segment of a circle containing an angle equal to the given angle, Euc. III. 33, and cutting EG in H; join HB, HD. HBD is the required triangle.

27. If the figure ABCD be one of four sides; join the opposite angles A, C of the figure, through D draw DE parallel to AC meeting BC produced in E, join AE:—the triangle ABE may be proved equal to the foursided figure ABCD.

If the figure ABCDE be one of five sides, produce the base both ways, and the figure may be transformed into a triangle, by two constructions similar to that employed for a figure of four sides. If the figure consist of six, seven, or any number of sides, the same process must be repeated.

28. From the given point A, let fall AB on the given straight line, and upon AB describe an equilateral triangle PAB. Produce AP to meet the given line in D, and bisect the angle BAD by AC meeting the given line in C. On AC, BD, describe the equilateral triangles QAC, RAD; if P, Q, R can be proved to be in a straight line the locus of the vertices of the triangles on the same side of AB will be a straight line.

29. Let ABC, be the given triangle whose base is BC. In BC, produced if necessary, take BD equal to the given base of the required isosceles triangle: on BD let BD be the base of the required isosceles triangle. On BD let a triangle BDE be formed equal to ABC. If this triangle be not isosceles, an equal triangle which shall be isosceles, may be formed on the same base BD, Euc. I. 37.

30. See Theorem 69, p. 304.

31. Let ABC be the required triangle having the right angle ABC, and such that the sum of AC and AB is double of BC.

Now since $AC + AB = 2BC$, therefore $AC - BC = BC - AB$. On AC take AD equal to this difference. Then $AC = BC + AD$, $AB = BC - AD$, and since $AC^2 = AB^2 + BC^2$, it follows that $BC^2 = 4BC.AD$, and $BC = 4AD$. Hence AD the difference between the hypothenuse and one side BC is known, and therefore the hypothenuse $AC = 5AD$. Hence the construction of the triangle depends on the division of the hypothenuse into five equal parts. See the remark on Prob. 2.

32. Let the two given lines meet in A, and let B be the given point.

If BC, BD be supposed to be drawn making equal angles with AC, AD and DC be joined, BCD is the triangle required, and the figure ACBD may be shewn to be a parallelogram. Whence the construction.

33. This is a particular case of Prob. 76, p. 351.

34. This problem cannot be solved without proportion.

35. This is a case of the more general problem:—To divide a triangle into two parts, having a given ratio to one another by a line parallel to any line given in position. See note on Prob. 76, p. 351.

36. Let D be the required point within the triangle ABC, such that the lines AD, BD, CD trisect the triangle. If these lines be produced to meet the sides of the triangle in E, F, G, it may be shewn that the sides are bisected in these points.

37. It is proved, Euclid I. 34, that each of the diagonals of a parallelogram bisects the figure, and it may easily be shewn that they also bisect each other. It is hence manifest that any straight line which bisects a parallelogram *must* pass

through the intersection of the diagonals. The different positions of a line through the intersection of the diagonals will suggest the constructions in the different cases.

38. (1) Reduce the trapezium ABCD to a triangle BAE by Prob. 27, and bisect the triangle BAE by a line AF from the vertex. If F falls without BC, through F draw FG parallel to AC or DE, and join AG.

Or thus. Draw the diagonals AC, BD; bisect BD in E, and join AE, EC. Draw FEG parallel to AC the other diagonal, meeting AD in F, and DC in G. AG being joined bisects the trapezium.

(2) Let E be the given point in the side AD. Join EB. Bisect the quadrilateral EBCD by EF. Make the triangle EFG equal to the triangle EBA, on that side of EF on which the greater part of ABCE lies. Bisect the triangle EFG by EH. EH bisects the figure.

39. (1) DF bisects the triangle ABC (fig. Prob. 3, p. 295). On each side of the point F in the line BC, take FG, FH, each equal to one third of BF, the lines DG, DH shall trisect the triangle. Or,

Let ABC be any triangle, D the given point in BC. Trisect BC in E, F. Join AD, and draw EG, FH parallel to AD. Join DG, DH: these lines trisect the triangle. Draw AE, AF and the proof is manifest.

(2) Let ABC be any triangle; trisect the base BC in D, E, and join AD, AE. From D, E, draw DP, EP parallel to AB, AC and meeting in P. Join AP, BP, CP; these three lines trisect the triangle.

(3) Let P be the given point within the triangle ABC. Trisect the base BC in D, E. From the vertex A draw AD, AE, AP. Join PD, draw AG parallel to PD and join PG. Then BGPA is one third of the triangle. The problem may be solved by trisecting either of the other two sides and making a similar construction.

40. Trisect the side AB in E, F, and draw EG, FH parallel to AD or BC, meeting DC in G and H. If the given point P be in EF, the two lines drawn from P through the bisections of EG and FH will trisect the parallelogram. If P be in FB, a line from P through the bisection of FH will cut off one third of the parallelogram, and the remaining trapezium is to be bisected by a line from P, one of its angles. If P coincide with E or F, the solution is obvious.

41. If a line be drawn from the given point in the side of the parallelogram through the intersection of the diagonals, the parallelogram is bisected; and the problem is reduced to the bisection of a trapezium two of whose sides are parallel, by a line drawn from the extremity of one of the parallel sides.

42. If a straight line be drawn from the given point through the intersection of the diagonals and meeting the opposite side of the square; the problem is then reduced to the bisection of a trapezium by a line drawn from one of its angles.

43. If the angles at the base of the isosceles triangle be bisected, the line joining the points where the bisecting lines meet the opposite sides of the triangle, will cut off the trapezium required.

44. The base may be divided into nine equal parts, and lines may be drawn from the vertex to the points of division. Or, the sides of the triangle may be trisected, and the points of trisection joined.

45. (1) By supposing the point P found in the side AB of the parallelogram ABCD, such that the angle contained by AP, PC may be bisected by the line PD; CP may be proved equal to CD; hence the solution is obvious.

(2) By supposing the point P found in the side AB produced, so that PD may bisect the angle contained by ABP and PC; it may be shewn that the side AB must be produced, so that BP is equal to BD.

46. Produce one side of the square till it become equal to the diagonal, the line drawn from the extremity of this produced side and parallel to the adjacent side of the square, and meeting the diagonal produced, determines the point required.

47. It is sufficient to suggest, that triangles on equal bases, and of equal altitudes, are equal.

48. Let ABC be the required triangle having the angle ACB a right angle. In BC produced, take CE equal to AC, and with centre B and radius BA describe a circular arc cutting CE in D, and join AD. Then DE is the difference between the sum of the two sides AC, CB and the hypothenuse AB; also one side AC the perpendicular is given. Hence the construction. On any line EB take EC equal to

the given side, ED equal to the given difference. At C, draw CA perpendicular to CB, and equal to EC, join AD, at A in AD make the angle DAB equal to ADB, and let AB meet EB in B. Then ABC is the triangle required.

49. Let AD be the sum of the base and hypothenuse, AB the sum of the base and perpendicular (figure, Euc. II. 10). At D draw a line perpendicular to AD, and through B draw EBG making with AB an angle ABE equal to half a right angle, and meeting the perpendicular to AD in G. Join AG and draw BH equal to BD and meeting AG in H. From A draw AE parallel to HB meeting EBG in E; draw EF parallel to AD meeting the perpendicular at D in F; and lastly, draw EC perpendicular to AD. Then the triangle AEC is the triangle required.

The proof of this problem involves the consideration of similar triangles; it should have been placed among the Problems to Book VI.

50. Draw two straight lines making an angle A equal to the given angle, and on one of them take a part AD equal to half the sum of the sides; bisect the angle A; then the problem is reduced to constructing a right-angled triangle, having given the sum of the base and the hypothenuse, and the angle contained between the perpendicular and the hypothenuse. Suppose the thing done, and that EF is the position of the base, join DF, and draw DG parallel to EF, the angle EDG is bisected by DF.

51. On the line which is equal to the perimeter of the required triangle describe a triangle having its angles equal to the given angles. Then the rest of the process is suggested by Prob. 2, p. 221.

52. Let ABC (fig. to Euc. I. 20) be the required triangle, having the base CB equal to the given base, the angle ABC equal to the given angle, and the two sides BA, AC together equal to the given line BD. Join DC, then since AD is equal to AC, the triangle ACD is isosceles, and therefore the angle ADC is equal to the angle ACD. Hence the construction of the triangle.

53. Let ABC be the required triangle (fig. to Euc. I. 18), having the angle ACB equal to the given angle, and the base BC equal to the given line, also CD equal to the difference of the two sides AB, AC. If BD be joined, then ABD is an isosceles triangle. Hence the synthesis. Does this construction hold good in all cases?

54. Let ABC be the required triangle, (fig. Euc. I. 18,) of which the side BC is given and the angle BAC, also CD the difference between the sides AB, AC. Join BD; then AB is equal to AD, because CD is their difference, and the triangle ABD is isosceles, whence the angle ABD is equal to the angle ADB; and since twice the angle ABD and BAD are equal to two right angles, it follows that ABD is half the supplement of the given angle BAC. Hence the construction of the triangle.

55. Let AB be the given base, at B draw BE at right angles to AB, with centre A and radius equal to the sum of the two remaining sides describe a circle cutting BE in E, and join AE. On BE take BF equal to the given altitude, and through F draw FC parallel to AB and meeting AE in C: join CB, then ACB is the triangle required.

56. Let CD be the given difference of the sides AB, AC (figure, Euc. I. 18), and BC the given base. Let ABC be the triangle required; at B draw BF perpendicular to BC, through A draw AE parallel to BC meeting BF in E, and produce CA to meet BF in F. Join BD. Then AD, AB, AF are equal to one another, and the triangle may be constructed on the base BC with altitude BE, and having the difference of the sides equal to CD. For the point D is determined by two circles which touch one another, one described with centre C and radius CD, and the other described passing through the points F, B, and touching the circle whose radius is CD in D. This problem requires the principles of Euc. III. for its construction.

57. Let ABC be the triangle, at C draw CD perpendicular to CB and equal to the sum of the required lines, through D draw DE parallel to CB meeting AC in E, and draw EF parallel to DC, meeting BC in F. Then EF is equal to DC. Next produce CB, making CG equal to CE, and join EG cutting AB in H. From H draw HK perpendicular to EAC, and HL perpendicular to BC. Then HK and HL together are equal to DC. The proof depends on Theorem 66, p. 303.

58. Let ABC, EBC, DBC (DB being joined) be three equal triangles on the same base BC and on the same side of it (fig. Euc. I. 41). Join AD, DE. Then AD is parallel to BC, and DE is parallel to BC.

59. The diameters of a square bisect one another at right angles.

60. This may be exhibited in different ways: one of the most simple, however, is the following. In the figure, Euc. II. 4. On DE, EB, take DE, EM, each equal to BC. Join CM and LM cutting GK in R and GF in Q; also join CH, HL, and draw MP parallel to BA, meeting GF in P. Then the square on CH is equal to the squares on HG, GC. The square on CG is divided into two parts by the line CR; and the square on HG into three parts by HL and LQ. The parts of the two squares HF, CK may be so arranged as to cover exactly the square CL.

61. Draw two indefinite lines AM, AN at right angles to each other. On AM take AB equal to the side of one of the given squares, and on AN take AC equal to the side of the second square, join BC, then the square on BC is equal to the squares on AB and AC. Again, on AM take AD equal to BC, and on AN take AE equal to the side of the third square; join DE, and the square on DE is equal to the three squares on AB, AC, AE.

62. The square described upon the diagonal of a square being equal to double the given square; a square may be described 8 times or $n$ times any given square, where $n$ is any power of the number 2.

63. This is an Algebraical Problem;—to find two rational numbers, the difference of whose squares is a given rational number.

Let the given base of the triangle contain ($a$) units, and if $x$, $y$, denote the hypothenuse and perpendicular; then $a^2 = x^2 - y^2 = (x+y)(x-y)$.

$$\text{Assume } x+y = na, \text{ and } x-y = \frac{1}{n}a:$$

$$\text{whence } x = \frac{(n^2+1).a}{2n}, \text{ and } y = \frac{(n^2-1).a}{2n}.$$

If $a = 2n$, then $2n$, $n^2+1$, $n^2-1$, are the three sides of the triangle, where $n$ is any integer greater than unity.

64. This problem is the same as to construct a right-angled triangle, having given the hypothenuse and one side.

65. If $x$, $y$ denote the base and perpendicular of the triangle, these values will be found to be 4 and 3 from the equations $x-y=1$, and $x^2+y^2=25$.

If a Geometrical Analysis be required. Let ABC be (fig. Euc. I. 18) the triangle required having the right angle at A, BC the given hypothenuse, and CD the given difference between the base AC and the perpendicular AB. Join BD, then BAD is an isosceles right-angled triangle. Since the angle ADB is half a right angle, and the two sides DC, CB are given, the point B can be found. Hence the synthesis.

66. See Theorem 70, p. 304.

67. First, let a parallelogram be formed on one side of the square, and having two of its sides of the required length; next, let a rhombus be formed on one of the sides of the required length. Euc. I. 35.

## HINTS, &c. TO THE THEOREMS. BOOK I.

2. Apply Euc. I. 6, 8.

3. This is proved by Euc. I. 32, 13, 5.

4. Let CAB be the triangle (fig. Euc. I. 10) CD the line bisecting the angle ACD and the base AB. Produce CD, and make DE equal to CD, and join AE. Then CB may be proved equal to AE, also AE to AC.

5. Construct the figure and apply Euc. I. 5, 32, 15.

If the isosceles triangle have its vertical angle less than two thirds of a right angle, the line ED produced meets AB produced towards the base and then 3.AEF = 4 right angles + AFE. If the vertical angle be greater than two thirds of a right angle, ED produced meets AB produced towards the vertex, then 3.AEF = 2 right angles + AFE.

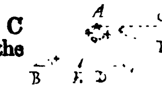
6. For in the figure Euc. I. 18, the two sides CB and BA are greater than CA, but AB is equal to AD, therefore the remainder BC is greater than DC or the difference between the two sides BA, AC of the triangle, is less than BC.

It may also be proved that the sum of the three sides of the triangle are *greater than* double any one of the sides, but *less than* the double of any two of the sides.

7. In the Theorem, for AC read BC. At C make the angle BCD equal to the angle ACB, and produce AB to meet CD in D.

8. If the given triangle have both of the angles at the base, acute angles; the difference of the angles at the base is at once obvious from Euc. I. 32. If one of the angles at the base be obtuse, does the property hold good? Yes.

9. Let ABC be a triangle, having the right angle at A, and the angle at C greater than the angle at B, also let AD be perpendicular to the base and AE be the line drawn to E the bisection of the base. AF ⊥ AF, AG [illegible]



Then AE may be proved equal to BE or EC independently of Euc. III. 31.

10. Let ABC be a triangle having the angle ACB double of the angle ABC, and let the perpendicular AD be drawn to the base BC. Take DE equal to DC and join AE. Then AE may be proved to be equal to EB.

If ACB be an obtuse angle, then AC is equal to the sum of the segments of the base made by the perpendicular from the vertex A.

11. By bisecting the hypothenuse, and drawing a line from the vertex to the point of bisection, it may be shewn that this line forms with the shorter side and half the hypothenuse an equilateral triangle. [illegible] It is difficult to me to prove this, as well as 9 directly.

12. Let the sides AB, AC of any triangle ABC be produced, the exterior angles bisected by two lines which meet in D, and let AD be joined, then AD bisects the angle BAC. For draw DE perpendicular on BC, also DF, DG perpendiculars on AB, AC produced, if necessary. Then DF may be proved equal to DG, and the squares of DF, DA are equal to the squares of FG, GA of which the square of FD is equal to the square of DG; hence AF is equal to AG, and Euc. I. 8, the angle BAC is bisected by AD.

13. Let ABC be the obtuse-angled triangle having the obtuse angle at A. Let the perpendiculars from D, E the bisections of AB, AC meet in G, join G and F the bisection of BC. If GF be proved perpendicular to BC, the theorem is proved.

NOTE. It may be more readily proved by transversals.

14. See Theorem 29, p. 321.

15. Constructing the figure, then by the Method of Transversals, D, E, G may be shewn to be in a straight line. See Cor. 3. Prop. III. Appendix, p. 22.

16. Let the two sides be produced and the exterior angles of the triangle be bisected: join the point in which the bisecting lines meet with the third interior angle of the triangle. If this line be proved to bisect the third interior angle of the triangle, the truth of the theorem is proved. (Conv. of Theorem 12, supra.)

17. This theorem is the converse of Theorem 10, supra.

18. The triangle ABE is proved equal to the triangle DCF, (fig. 2.) In FD if FG be taken equal to ED, and GH be drawn parallel to DC, the triangle FGH is equal to the triangle whose base is DE.

19. Let AD bisect the base, and AE the vertical angle A, and meeting the base in the points D, E. The angle AED may be shewn to be greater than the angle ADE.

20. Let ABC be the triangle; AD perpendicular to BC, AE drawn to the bisection of BC, and AF bisecting the angle BAC. Produce AD and make DA′ equal to AD: join FA′, EA′.

21. In the figure, Euc. I. 47, FC is always equal to AD, and AE to BK. If AB be equal to AC, the truth of the proposition is manifest.

22. This theorem is misplaced, as it cannot be proved by the first book.

First. Prove that the perpendiculars A*a*, B*b*, C*c* pass through the same point O, as Theo. 29, p. 321; or by the theory of Transversals, Prop. II. Appendix, p. 22. Secondly. That the triangles A*cb*, B*ac*, C*ab* are equiangular to ABC. Euc. III. 21. Thirdly. That the angles of the triangle *abc* are bisected by the perpendiculars; and lastly, by means of Prob. I. p. 293, that $ab+bc+ca$ is a minimum.

23. Let FC be perpendicular to AB and FE be drawn to any other point E in AB: then FC may be proved less than FE, Euc. I. 32, 19. Let CD be taken equal to CE and FD be joined, then FD is equal to FE, and no other line can be drawn from F equal to FE; if possible let FA be equal to FE, which may be shewn to be impossible.

24. Let a circle be described through the points A, B, C; bisect AB in D and

draw the diameter EDF; then the line which bisects the angle C may be proved to pass through the point F on the other side of AB. Euc. III. 21.

25. This proposition requires for its proof the case of equal triangles omitted in Euclid:—namely, when two sides and one angle are given, but not the angle included by the given sides.

26. Let A be the given point, and BC, BD the two straight lines intersecting each other in B. Suppose AEF the line required, such that AF is terminated by BD, and bisected in E by BC. Join AB, draw AG parallel to BD, and join GF. Then ABFG is a parallelogram.

27. This is Prop. 33, of Euclid's Data. There are obviously two points in the given line to which lines may be drawn from the given point.

28. This may very easily be shewn. A restriction however is necessary—namely, that the angles of the interior figure are turned *from* the base.

29. This is possible in certain cases. As an instance, a right-angled triangle ABC may be taken having the right angle at B. From A draw any line AD to meet the base BC in D. Take DE equal to AB; bisect AE in F, and join FC; then the sum of the lines CF, FD shall be greater than the sum of CA, AB, the sides of the triangle. Pappi Coll. Math. Lib. III. Prop. 28.

30. See the notes on Euc. I. 29, p. 50; also, Appendix, p. 2.

31. See the notes on Euc. I. 29, p. 50; and Appendix, p. 2.

32. This is manifest from Euc. I. 29.

33. This will appear from Euc. I. 29, 15, 26.

34. Draw the diagonal and apply Euc. I. 8, 28. The figure is either a square or a rhombus.

35. This is only a more general case of the last Proposition.

36. Apply Euc. I. 29, 26.

37. If the square and parallelogram be upon the same base and between the same parallels, the truth is obvious from Euc. I. 37.

38. The former assertion is proved from Euc. I. 29, 26. The latter may be shewn indirectly.

39. This is proved by applying Euc. I. 8, 4.

40. Let fall upon the diagonal perpendiculars from the opposite angles of the parallelogram. These perpendiculars may be proved to be equal, and each pair of triangles is situated on different sides of the same base and has equal altitudes.

41. Let the line drawn from A fall without the parallelogram, and let CC′, BB′, DD′ be the perpendiculars from C, B, D, on the line drawn from A; from B draw BE parallel to AC′, and the truth is manifest. Next, let the line from A be drawn so as to fall within the parallelogram.

42. Let FH, GE (figure, Euc. I. 43) be joined and produced, they will meet the diagonal CA produced in the same point L. The lines CA, GE, FH may be proved by similar triangles *to converge* to one point, and when produced, to meet in that point. This theorem properly falls under the theorems on Euc. VI.

43. The perpendiculars drawn from B and C are to be perpendicular to the sides AB, AC respectively. Let ABDC be the parallelogram, DE perpendicular on BC the diagonal. At B let BF be drawn meeting ED, produced if necessary in F. Join F, C. If FC can be shewn to be perpendicular to AC, the theorem is proved.

44. One case of this theorem is included in Theorem 40, supra. The other case, when the point is in the diagonal produced, is obvious from the same principle.

45. Let ABCD be a parallelogram and P any point without it, and AC the diagonal. Let AP, PD, PB, PC be joined. Then the triangles APD and APB together are equivalent to the triangle APC. Draw PGE parallel to AD meeting AB and DC in G, E, and join DG, CG. Then by Euc. I. 37.

46. If the four sides of the figure be of different lengths, the truth of the theorem may be shewn. If, however, two adjacent sides of the figure be equal to one another, as also the other two: the lines drawn from the angles to the bisection of the longer diagonal, will be found to divide the trapezium into four triangles which are equal in area to one another.

47. Let BCED be a trapezium (fig. Euc. VI. 2.) of which DC, BE are the diagonals intersecting each other in G. If the triangle DGB be equal to the triangle EGC, the side DE may be proved parallel to the side BC, by Euc. I. 39.

48. Through the point of bisection of one of the opposite sides which are not parallel, draw a line parallel to the opposite side, and meeting the parallel sides, produced, if necessary.

49. This may be shewn by Euc. I. 20.

50. This may be shewn by Euc. I. 35.

51. Draw the two diagonals, then four triangles are formed, two on one side of each diagonal. Then two of the lines drawn through the points of bisection of two sides may be proved parallel to one diagonal, and two parallel to the other diagonal, in the same way as Theorem 45, (which ought to have been placed earlier). The other property is manifest from the relation of the areas of the triangles made by the lines drawn through the bisections of the sides.

52. Join A′B′, B′C′, C′D′, D′A′; then the triangles D′AA′, B′CC′, may be proved to be equal in all respects; as also the triangles A′BB′, C′DD′: whence the figure A′B′C′D′ may be proved to be a parallelogram.

53. See the fig. Euc. VI. 2. The triangle ABC is double of the triangle ABE, and the triangle ABE is double of the triangle ADE. Hence the triangle ADE is one fourth of the triangle ABC. The line DE which bisects the sides may easily be shewn to be parallel to BC.

54. The lines bisecting the opposite sides of a trapezium may be shewn to be the diagonals of the parallelogram formed by joining the four points of bisection of the sides of the trapezium.

55. Let the figure be constructed, and let AC, BD, intersect each other in E. Then by Euc. I. 6, 15, 26, 4, 5, 32, 29.

56. If the *isosceles triangle* be obtuse-angled, by Euc. I. 5, 32, the truth will be made evident. If the triangle be acute-angled the enunciation of the proposition requires some modification.

57. Let ED be bisected in F, and join AF, then AF is equal to EF, and by Euc. I. 5, 29, 32, the angle ABD is proved to be double the angle DBC.

58. The points A, C are two points in the adjacent sides BF, BD produced of the parallelogram. It may be shewn that so long as the figure BFED is a parallelogram, the angles made by FE, DE at the point E, with AE and CE, are together equal to two right angles, and therefore by Euc. I. 14, the line AE is in the same straight line with EC.

In the enunciation for "points A, G and C," read "points A, E and C."

59. The most direct method of shewing that the three other diagonals (which bisect the sides of the triangle) pass through the same point, is by means of transversals.

60. Let ABC be an isosceles triangle (fig. Euc. I. 42), AE perpendicular to the base BC, and AECG the equivalent rectangle. Then AC is greater than AE, &c.

61. This is a case of the preceding theorem.

Of all the triangles which are equal to one another, the isosceles has the least perimeter, and it is easily shewn that the perimeter of the triangle or twice the side and diagonal of the square is greater than the perimeter of the square.

62. Let the angles at the base BC be acute angles. Join DE, CD, CE. Then C is a point within the parallelogram DABE, and the triangles ACD, BCE may be shewn to be together equal to half the parallelogram or double of the triangle ABC.

63. If ABC, DBC be two equal triangles on the same base, of which ABC is isosceles, fig. Euc. I. 37. By producing AB and making AG equal to AB or AC and joining GD, the perimeter of the triangle ABC may be shewn to be less than the perimeter of the triangle DBC.

64. Let two triangles be constructed on the same base with equal perimeters, of which one of them is isosceles. Through the vertex of that which is not isosceles draw a line parallel to the base, and intersecting the perpendicular drawn from the vertex of the isosceles triangle upon the common base. Join this point of intersection and the extremities of the base.

65. Let ABC be a triangle whose vertical angle is A, and whose base BC is bisected in D; let any line EDG be drawn through D, meeting AC in G and AB produced in E, and forming a triangle AEG having the same vertical angle A. Draw BH parallel to AC, and the triangles ADH, GDC are equal. Euc. I. 26.

66. Let ABC be an isosceles triangle, and from any point D in the base BC,

and the extremity B, let three lines DE, DF, BG be drawn to the sides and making equal angles with the base. Produce ED and make DH equal to DF and join BH.

67. Let fall also a perpendicular from the vertex on the base.

68. In the fig. Euc. I. 1, produce AB both ways to meet the circles in D and E, join CD, CE, then CDE is an isosceles triangle, having each of the angles at the base one fourth of the angle at the vertex. At E draw EG perpendicular to DB and meeting DC produced in G. Then CEG is an equilateral triangle.

69. On the same base AB, and on the same side of it, let two triangles ABC, ABD be constructed, having the side BD equal to BC, the angle ABC a right angle, but the angle ABD not a right angle; then the triangle ABC may be shewn to be greater than the triangle ABD whether the angle ABD be acute or obtuse.

70. Let any parallelograms be described on any two sides AB, AC of a triangle ABC, and the sides parallel to AB, AC be produced to meet in a point P. Join PA. Then on either side of the base BC, let a parallelogram be described having two sides equal and parallel to AP. Produce AP and it will divide the parallelogram on BC into two parts respectively equal to the parallelograms on the sides. Euc. I. 35, 36.

71. (*a*) Apply Euc. I. 5, 29.

(*b*) The question is imperfectly expressed.

If FC intersect AE in $p$ and AB in Q; also BK intersect AD in $q$ and AC in P: then A$p$ is equal to P$q$ and A$q$ to Q$p$. In the case only of AB being equal to AC will the parts A$p$, A$q$ cut off from AE, AD be equal to one another.

(*c*) Let AL meet the base BC in P, and let the perpendiculars from F, K meet BC produced in M and N respectively: then the triangles APB, FMB may be proved to be equal in all respects, as also the triangles APC, CKN.

(*d*) If FH, FG be produced and meet in O, and OB, OC be joined, the triangle OBC is equal in all respects to the triangle ADE, and joining the points A, O, the line AO is in the same straight line with AL which meets the base BC in R. Then OR is a perpendicular from an angle on the opposite side BC of the triangle OBC. If BK, CF can be proved to be perpendicular to the other two sides OC, OB respectively: then BK and CF intersect AL in the same point.

(*e*) Let fall DQ perpendicular on FB produced. Then the triangle DQB may be proved equal to each of the triangles ABC, DBF; whence the triangle DBF is equal to the triangle ABC.

Perhaps however the better method is to prove at once that the triangles ABC, FBD are equal, by shewing that they have two sides equal in each triangle, and the included angles, one the supplement of the other.

(*f*) If DQ be drawn perpendicular on FB produced, FQ may be proved to be bisected in the point B, and DQ equal to AC. Then the square of FD is found by the right-angled triangle FQD. Similarly, the square of KE is found, and the sum of the squares of FD, EK, GH will be found to be six (not eight) times the square of the hypothenuse; but the sum of the squares of FD, DE, EK, KH, HG, GF are together equal to eight times the square of the hypothenuse.

(*g*) The three triangles formed by joining GH, KE, DF are each equal to the triangle ABC. Whence the sum of the four triangles and the three squares may be shewn to be equal to the sum of two squares, namely of BC and of AB + AC.

72. The former part is at once manifest by Euc. I. 47. Let the diagonals of the square be drawn, and the given point be supposed to coincide with the intersection of the diagonals, the minimum is obvious. Find its value in terms of the side.

73. Apply Euc. I. 47.

74. Let the base BC be bisected in D, and DE be drawn perpendicular to the hypothenuse AC. Join AD: then Euc. I. 47.

75. This is at once obvious from Euc. I. 47.

76. Let the equilateral triangles ABD, BCE, CAF be described on AB, BC, CA the sides respectively of the triangle ABC having the right angle at A.

Join DC, AK: then the triangles DBC, ABE are equal. Next draw DG perpendicular to AB and join CG: then the triangles BDG, DAG, DGC are equal to one another. Also draw AH, EK perpendicular to BC; the triangles EKH, EKA are equal. Whence may be shewn that the triangle ABD is equal to the triangle BHE, and in a similar way may be shewn that CAF is equal to CHE.

The restriction is unnecessary: it only brings AD, AE into the same line.

## HINTS, &c. TO THE PROBLEMS. BOOK II.

2. Prob. I. p. 306, suggests the method to be employed.

3. Let AB be the given straight line (fig. Euc. II. 4). On AB describe a square, draw the diagonal BD, and take AC equal to half the diagonal BD.

4. Let BF be the given line, (fig. Euc. II. 14) and suppose E the point of division, such that the rectangle BE, EF is equal to the square of the difference of BE and EF. On FB describe the semicircle BHF and draw EH perpendicular to BF to meet the circumference in H. Join G, H, and produce GH to meet in K, the line FK drawn perpendicular to BF. Then FK may be shewn to be equal to FB. Hence the construction.

5. A square may be found equal to the given rectangle; and then the Prob. is reduced to Prob. I. p. 306.

6. This is a repetition of Prob. 2, by mistake.

7. Let AB be the given line. Find a line AE of which the square shall be three times the square of AB; from AB cut off AC equal to the difference between AE and AB, and C is the point of section such that the squares of AB and BC are double the square of AC.

If the square of one part were required to be three times the square of the other, the problem, by aid of Euc. II. 7, is at once reduced to depend on Euc. II. 11.

8. From the preceding Problem $AB^2 + BC^2 = 2AC^2$;

$$\text{therefore } AB^2 - AC^2 = AC^2 - BC^2;$$
$$\text{or } (AB + AC).(AB - AC) = (AC + BC).(AC - BC);$$
$$\text{or } (AB + AC).BC = AB.(AC - BC);$$
$$\text{or } AB.BC + AC.BC = AB.AC - AB.BC,$$
$$\text{therefore } 2.AB.BC = AC.(AB - BC) = AC^2.$$

Whence the problem depends upon the preceding.

9. See note on Euc. II. 11, p. 72.

10. By assuming the points of division, it will be found that the line must be so divided that the square of the middle part is equal to twice the rectangle contained by the extreme parts. Let AB be the given line. Describe on AB any right-angled triangle not isosceles. With centre B and radius BC describe a circle cutting AB in D; and with centre A and radius AC describe a circle cutting AB in E. Then D, E, are the points required.

11. Let the point C be supposed to be determined; then since the rectangle of the sum and difference of AB and BC is equal to the difference of the squares of AB and BC, of which AB is known and BC unknown, which difference is equal to the square of a given line AC. Hence BC is known by Euc. I. 47.

12. Let AB the given straight line be divided in C, it is required to produce AB so that the rectangle contained by the whole line produced and the part produced, may be equal to the rectangle contained by AB, AC. Find by Euc. II. 14, the side of a square which is equal to the rectangle contained by AB, AC: and then the problem is reduced to that of producing the line AB to some point D, so that the rectangle contained by AD, DB is equal to a given square.

13. This follows more simply from Euc. III. 36. If BD be the side of the given square, and AC the difference of the adjacent sides of the rectangle.

14. Let EH be the side of the given square (fig. Euc. II. 14), and BF the sum of the adjacent sides—the construction is obvious.

15. Euc. I. 45, 47; II. 14; will suggest the necessary constructions.

16. Let ABCD be the given rectangle. From A draw AB′ equal to the given length to meet DC, or DC produced in B′; draw BE perpendicular to AB′, and upon the other side of AB′ describe a rectangle AB′C′D′ having AD′ equal to BE. The line C′D′ may cut DC, or DC produced, either way. If D′C′ cut AD in F so that AF is less than FD, then FG must be taken equal to AF and a line be drawn through G parallel to D′C′ to meet DC. The two rectangles may be shewn to consist of parts common or mutually equal to each other.

17. There seems to be some inaccuracy in the enunciation of this Problem. In its present form it appears to be impossible, except when C is the middle of AB, and D coincides with it.

18. In the question for $4c$ read $2c$; otherwise the problem is impossible, Euc. I. 19. To construct the problem generally:—

Assume any line AB to represent $c$; on AB describe a square ABCD; produce AB to E. On AE describe a semicircle intersecting CD in F. Draw FG perpendicular to AE. Then AG, AB, GE are the sides of the triangle. If BE be equal to AB, F coincides with C, and the triangle is equilateral.

## HINTS, &c. TO THE THEOREMS. BOOK II.

3. The area of the rhombus is equal to the areas of the four *right-angled triangles* formed by the diagonals and sides of the figure.

4. Let ABCD be a trapezium having the side AB parallel to CD. Draw the diagonal AC; then the area of the trapezium is equal to the two triangles. See note on Euc. II. def. p. 68.

5. Apply Euc. II. 5, Cor. and note p. 68.

6. In the figure, Euc. II. 7. Join BF, and draw FL perpendicular on GD. Half the rectangle DB, BG, may be proved equal to the rectangle AB, BC. Or,

Join KA, CD, KD, CK. Then CK is perpendicular to BD. And the triangles CBD, KBD are each equal to the triangle ABK. Hence, twice the triangle ABK is equal to the figure CBKD; but twice the triangle ABK is equal to the rectangle contained by AB, BC; and the figure CBKD is equal to half the rectangle contained by DB and CK, the diagonals of the squares on AB, BC. Wherefore, &c.

7. This follows from Euc. II. 7.

8. The difference between the two unequal parts may be shewn to be equal to twice the line between the points of section.

9. This proposition is only another form of stating Euc. II. 7.

10. This may be shewn from Euc. II. 5, Cor.

11. See the notes on Euc. II. 5, 6, 10, 11, p. 69, &c.

12. The Problem is, in other words, given the sum of two lines and the sum of their squares, to find the lines.

Take AB equal to the side of the given square, and on AB describe a semicircle ADB, D being the middle point. Join DA, DB. With centre D and distance DA or DB describe a circle AEB; and with centre A and radius equal to the sum of the given lines, describe another circle cutting AEB in E. Draw AE cutting ADB in C, and join CB. AC, CB are the lines required.

13. Through E draw EG parallel to AB, and through F, draw FHK parallel to BC and cutting EG in H. Then the area of the rectangle is made up of the areas of four triangles; whence it may be readily shewn that *twice the area* (not the area) of the triangle AFE, and the figure AGHK is equal to the area of the rectangle.

14. The lines must be reckoned *positive* or *negative*, according to the direction in which they are measured; and the theorem is rather algebraical: From

$$AF^2=(AE+EF)^2;\ BF^2=(BE+EF)^2;\ CF^2=(EF-CE)^2;\ DF^2=(EF-DE)^2,$$

the enuncicated property may be found.

15. Draw lines from the angles of the equilateral triangle to the point from which the perpendiculars are drawn to the sides. The equilateral triangle is divided into three triangles, the sum of whose areas are equal to the area of the whole triangle. Euc. I. 41; II. def. 1. The point may next be supposed to fall on one of the sides, and the consequence remarked; and lastly, outside the triangle.

16. In the question, for C read H, as in the figure, Euc. II. 11. If D be the point in AH, so that $HD = BH$, then $AB = AH + BH$; and since $AB . BH = AH^2$,

$$\therefore (AH+BH).BH = AH^2, \text{ and } \therefore BH^2 = AH^2 - AH.BH = AH.(AH-BH);$$

$$\text{or, } HD^2 = AH.AD; \text{ that is, AH is divided in D,}$$

so that the rectangle contained by the whole line and one part, is equal to the squrre

of the other part. By a similar process, HD may be so divided; and so on, by always taking from the greater part of the divided line, a part equal to the less.

Also, if BA be produced, and AK be taken equal to AB, KM to KH, ML to MA, and so on; KH, MA, LK, &c. may be proved to be similarly divided to AB. The proof that KH is so divided appears from the figure: for $CF . FA = CA^2$; and $AK = AC$, also $AF = AH$.

17. The succession of steps may be traced through the first and second books, the final step being Euc. II. 14.

18. From C let fall CF perpendicular on AB. Then ACE is an obtuse-angled triangle, and BEC is an acute-angled triangle. Apply Euc. II. 12, 13; and by Euc. I. 47, the squares of AC and CB are equal to the square of AB.

19. Let ABDE be the square on AB; from C draw CF, CG perpendiculars on DB, EA produced. Then by Euc. II. 12.

20. In a right-angled triangle (Euc. I. 47), the square of the side subtending the right angle is *equal to* the squares of the sides containing the right angle; but in an obtuse-angled triangle (Euc. II. 12), the square of the side subtending the obtuse angle *is greater than* the square of the side containing the obtuse angle; and in an acute-angled triangle (Euc. II. 13), the square of the side subtending an acute angle *is less than* the squares of the sides which contain that angle.

21. This will be found to be that particular case of Euc. II. 12, in which the distance of the obtuse angle from the foot of the perpendicular is half of the side subtended by the right angle made by the perpendicular and the base produced.

22. In every scalene triangle, the line drawn from the vertex to the bisection of the base, divides the triangle into two triangles, one obtuse-angled and the other acute-angled. Apply Euc. II. 12, 13.

23. (1) Let the triangle be acute-angled (Euc. II. 13, fig. 1.)

Let AC be bisected in E, and BE be joined; also EF be drawn perpendicular to BC. DF is equal to FC. Then the square of BE may be proved to be equal to the square of EC together with the rectangle BD, BC.

(2) If the triangle be obtuse-angled, the perpendicular EF falls *within* or without the base according as the bisecting line is drawn from the *obtuse* or the *acute* angle at the base.

24. See Theorem 29, p. 354.

25. The truth of this theorem follows at once from Euc. I. 47.

26. The common intersection of the three lines divides each into two parts, one of which is double of the other, and this point is the vertex of three triangles which have lines drawn from it to the bisection of the bases. Apply Euc. II. 12, 13.

27. Draw a perpendicular from the vertex to the base, and apply Euc. I. 47; II. 5. Cor. Enunciate and prove the proposition when the straight line drawn from the vertex meets the base produced.

28. This follows directly from Euc. II. 13, Case 1.

29. The truth of this proposition may be shewn from Euc. I. 47; and Euc. II. 4.

30. Let the square on the base of the isosceles triangle be described. Draw the diagonals of the square, and the proof is obvious.

31. Apply Euc. I. 47, to express the squares of the three sides in terms of the squares of the perpendicular and of the segments of AB.

32. Draw EF parallel to AB and meeting the base in F: draw also EG perpendicular to the base. Then DF is a parallelogram, and by Euc. I. 47; II. 5. Cor.

33. Bisect the angle B by BD meeting the opposite side in D, and draw BE perpendicular to AC. Then by Euc. I. 47; II. 5. Cor.

34. This follows directly from Theorem 22, p. 309.

35. Let the point O be within the rectangle. Draw the diagonals intersecting each other in P and join OP. Euc. II. 12, 13. Let O be without the triangle.

36. Draw from any two opposite angles, straight lines to meet in the bisection of the diagonal joining the other angles. Then by Euc. II. 12, 13.

37. Draw two lines from the point of bisection of either of the bisected sides to the extremities of the opposite side; and three triangles will be formed, two on one of the bisected sides and one on the other, in each of which is a line drawn from the vertex to the bisection of the base. Then by Theorem 22, p. 309.

38. If the extremities of the two lines which bisect the opposite sides of the

trapezium be joined, the figure formed is a parallelogram which has its sides respectively parallel to, and equal to, half the diagonals of the trapezium. The sum of the squares of the two diagonals of the trapezium may be easily shewn to be equal to the sum of the squares of the four sides of the parallelogram.

39. Draw perpendiculars from the extremities of one of the parallel sides, meeting the other side produced, if necessary. Then from the four right-angled triangles thus formed, may be shewn the truth of the proposition.

40. Let ABC be any triangle; AHKB, AGFC, BDEC, the squares upon their sides; EF, GH, KL the lines joining the angles of the squares. Produce GA, KB, EC, and draw HN, DQ, FR perpendiculars upon them respectively: also draw AP, BM, CS perpendiculars on the sides of the triangle. Then AN may be proved to be equal to AM; CR to CP; and BQ to BS; and by Euc. II. 12, 13.

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## HINTS, &c. TO THE PROBLEMS. BOOK III.

3. Let A be the centre, the radius of the circle being unknown. Take any point B in the circumference and find by means of the compasses only, a third point C in the circumference, such that B, A, C, shall be in the same straight line.

Many problems of this class are solved in the "*Géométrie du Compas,*" *par L. Mascheroni.*

4. Euc. III. 3, suggests the construction.

5. The least chord drawn through a given point, is the line perpendicular to that diameter which passes through the given point.

6. The given point may be either within or without the circle. Find the centre of the circle, and join the given point and the centre, and upon this line describe a semicircle, a line equal to the given distance may be drawn from the given point to meet the arc of the semicircle. When the point is without the circle, the given distance may meet the diameter produced.

7. Let two unequal circles cut one another, and let the line ABC drawn through B, one of the points of intersection, be the line required, such that AB is equal to BC. Join OO′ the centres of the circles, and draw OP, O′P′ perpendiculars on ABC, then PB is equal to BP′; through O′ draw O′D parallel to PP′; then ODO′ is a right-angled triangle, and a semicircle described on OO′ as a diameter will pass through the point D. Hence the synthesis. If the line ABC be supposed to move round the point B and its extremities A, C to be in the extremities of the two circles, it is manifest that ABC admits of a maximum.

8. It is sufficient to suggest that the angle between a chord and a tangent is equal to the angle in the alternate segment of the circle. Euc. III. 32.

9. There is some inaccuracy in the enunciation:—for if the two points be given in the circumference of the circle, the angle which the tangents make with the given line is dependent on the position of the two points in the circumference.

10. Let D be the point required in the diameter BA produced, such that the tangent DP is half of DB. Join CP, C being the centre. Then CPD is a right-angled triangle, having the sum of the base PC and hypothenuse CD double of the perpendicular PD. See Prob. 31, p. 298.

11. Let P be the given point, and PBA the given line cutting the circle ABC in the points B, A. Let PCD be the line required: join OC, OD, O being the centre. Then the arc AB being given, and the sum of the arcs BC, AD; the arc CD is also given in magnitude, and the angle COD which it subtends at the centre.

Whence the construction. Take the arc RS equal to the defect of the sum of the three arcs AB, DA, BC from the whole circumference: join RS, and with centre O describe a circle touching RS, and from P draw PCD to touch this circle.

12. At any point P in the circumference of the given circle, draw a line APB touching the circle at P, the parts AP, BP being equal to the given line. With centre C and radius CA or CB describe a circle, produce the given chord DE to meet the circumference of this circle in F. From F draw FG to touch the given circle in G, then FG is the line required.

14. Describe a circle through the three given points; from A any one of them, draw any chord, and from the centre D draw the perpendicular DE upon it. With the same centre and radius DE describe a circle. Then from B, C draw lines BF, CG touching this circle; then AE, BF, CG are equal to one another. The circle, which is the superior limit, is obviously the circle passing through the points A, B, C.

15. Find the centre, and the construction of Euc. III. 15 will suggest how the radius may be found, supposing FG and BC on the same side of the diameter.

16. The lengths of the lines may be found by Euc. III. 15.

17. Let ABC be the required triangle on the given base AB, having the sum of the squares of its sides AC, CB equal to the given square. If the base AB be bisected in D and CD be joined, then by Theo. 22, p. 309, the difference between the sum of the squares of the sides and of the square of half the base may be shewn to be equal to double the square of CD. Hence CD is constant, and therefore the locus is a circle whose centre is D and radius DC.

18. Let O be the centre of the given circle. Draw OA perpendicular to the given straight line: at O in OA make the angle AOP equal to the given angle, produce PO to meet the circumference again in Q. Then P, Q are two points from which tangents may be drawn fulfilling the required condition.

19. (1) When the tangent is on the same side of the two circles. Join C, C′ their centres, and on CC′ describe a semicircle. With centre C′ and radius equal to the *difference* of the radii of the two circles, describe another circle cutting the semicircle in D; join DC′ and produce it to meet the circumference of the given circle in B. Through C draw CA parallel to DB and join BA; this line touches the two circles.

(2) When the tangent is on the alternate sides. Having joined C, C′; on CC′ describe a semicircle; with centre C, and radius equal to the *sum* of the radii of the two circles describe another circle cutting the semicircle in D, join CD cutting the circumference in A, through C draw CB parallel to CA and join AB.

20. Let A be the given point, PQ any quadrantal arc of the given circle. From C the centre draw CD perpendicular to PQ and with centre C and radius CD, describe a circle, which will touch PQ in D. Then two tangents may be drawn to this circle from A, either of which is a solution. The same construction serves for any given chord *less than* the diameter of the given circle.

21. Let AB, BC, CD be the three given straight lines in the same straight line. On AC as a base describe the locus of the vertex of the triangle whose sides are as AB to BC. On the base BD describe the locus of the vertex of the triangle whose sides shall be as BC to CD. Let P be the intersection of these loci. Join PA, PB, PC, PD; the angles ABP, BPC, CPD are equal angles. See Prob. 8, p. 348.

22. Let A be the given point in the diameter BC; through A draw DAE perpendicular to BC, and join DB, BE. Through A draw any other chord FAG; and join BF, BG; draw FH, GK parallel to BC meeting DE in H, K respectively; join BH, BK, and draw FM, GN perpendicular to BC. Then HK may be proved to be less than DE, and the triangle FBG less than the triangle DBA.

23. From A suppose ACD drawn, so that when BD, BC are joined, AD and DB shall together be double of AC and CB together. Then the angles ACD, ADB are supplementary, and hence the angles BCD, BDC are equal, and the triangle BCD is isosceles. Also the angles BCD, BDC are given, hence the triangle BDC is given in species.

Again $AD+DB=2.AC+2.BC$, or $CD=AC+BC$.

Whence, make the triangle $bdc$ having its angles at $d$, $c$ equal to that in the segment BDA; and make $ca=cd-cb$, and join $ab$. At A make the angle BAD equal to $bad$, and AD is the line required.

24. This is the same as Problem 12, p. 315.

25. Let the given straight line AB be divided at any point C. On AB as a diameter describe a circle. Let the point D in the arc be the point required, such that when DA, DC, DB are joined, the angles ADC, CDB are each equal to half a right angle. Produce DC to meet the circumference in E, then Euc. III. 21.

26. The line which divides the circle into two segments is equal to the line which joins the two points of intersection of the circles in the figure Euc. I. 1.

27. Let ABC be a triangle of which the base or longest side is BC, and let a segment of a circle be described on BC. Produce BA, CA to meet the arc of the

segment in D, E, and join BD, CE. If circles be described about the triangles ABD, ACE, the sides AB, AC shall cut off segments similar to the segment described upon the base BC.

28. This is the same as Euc. III. 34, with the condition, that the line must pass through a given point.

29. By supposing the thing done, the line joining the points of intersection of the two circles, will pass through the centre of the given circle. From one of the given points draw a line through the centre of the given circle and cutting the two circles in four points; then by Euc. III. 35.

30. As a particular case, if the segment be a semicircle: on AB the diameter, the point K will be found to be distant from B, one half of the radius.

31. If a segment of a circle containing an angle equal to the given angle be described upon the line which joins the two given points, the circumference of the segment will meet the circumference of the given circle in two points, which will give two solutions of the Problem; or in one point at which the angle will be a maximum.

32. Let ABCD be the required trapezium inscribed in the given circle (fig. Euc. III. 22) of which AB is given, also the sum of the remaining three sides and the angle ADC. Since the angle ADC is given, the opposite angle ABC is known, and therefore the point C and the side BC. Produce AD and make DE equal to DC and join EC. Since the sum of AD, DC, CB is given, and DC is known, therefore the sum of AD, DC is given, and likewise AC, and the angle ADC. Also the angle DEC being half of the angle ADC is given. Whence the segment of the circle which contains AEC is given, also AE is given, and hence the point E, and consequently the point D. Whence the construction.

33. Join the centres A, B; at C the point of contact draw a tangent, and at A draw AF cutting the tangent in F, and making with CF an angle equal to one fourth of the given angle. From F draw tangents to the circles.

34. At any points P, R in the circumferences of the circles whose centres are A, B, draw PQ, RS, tangents equal to the given lines, and join AQ, BS. These being made the sides of a triangle of which AB is the base, the vertex of the triangle is the point required.

35. Let C, C′ be the centres of the given circles, join CC′, and let DD′ be the line required, making the given angle with CC′. Through C draw CE making with CC′ an angle equal to the given angle, and equal to the given line, join CD, ED′.

36. Let ABCD be the required line, such that the chords AB, CD are each equal to the given line. Take O, O′ the centres of the circles, and draw OE perpendicular to AB, and O′F to CD. With centres O, O′ and radii OE, O′F respectively, describe two circles. Then the line ABCD is a common tangent to these circles, and all chords which touch the interior and are terminated by the exterior circles are equal to one another.

37. Suppose the thing done, then it will appear that the line joining the points of intersection of the two circles is bisected at right angles by the line joining the centres of the circles. Since the radii are known, the centres of the two circles may be determined.

38. Let the two circles cut one another in A and B. Join AB, and suppose ACD to be the line required meeting the circumferences in C, D, such that the part DE is equal to the given line. Join also BC, BD. Then the angles at C, D may be shewn to be given, and the point D. The proof of this problem involves proportion.

39. Let A, B, C be the three given points. Join A, B, and on AB describe a segment of a circle ADB containing an angle equal to that which the lines drawn to the points A, B is to contain, and complete the circle. On the other side of AB at the point B, make the angle ABE equal to the angle which the lines drawn to the points A, C, is to contain, and let the line BE meet the circumference in E. Join EC and produce it to meet the circumference again in D. D is the point required.

40. Let the two circles touch each other at A, AD, AD′, their diameters; and let PQM be the semichord perpendicular to the diameter. If PQ be supposed equal to QM, the line MD may be shewn to be equal to four times MD′.

41. Let the straight line joining the centres of the two circles be produced both ways to meet the circumference of the exterior circle.

42. Let A be the common centre of the two circles, and BCDE the chord such that BE is double of CD. From A, B draw AF, BG perpendicular to BE. Join AC, and produce it to meet BG in G. Then AC may be shewn to be equal to CG, and the angle CBG being a right angle, is the angle in the semicircle described on CG as its diameter.

43. Let C be the common centre. Draw any diameter BCD of the interior circle, and produce it, making DE equal to CD. On DE describe a semicircle cutting the circumference of the exterior circle in F, join FD and produce it to meet the interior circumference in G and the exterior in H.

44. It may be proved that of all triangles having equal vertical angles, and their distances from their vertices to the bisections of their bases, equal to one another, the greatest is that which is isosceles.

45. The locus of the point C when the string ACB is kept stretched is an ellipse. This Problem appears here by mistake.

46. Let A be the given point (fig. Euc. III. 36, Cor.) and suppose AFC meeting the circle in F, C, to be bisected in F, and let AD be a tangent drawn from A. Then $2.AF^2 = AF.AC = AD^2$, but AD is given, hence also AF is given.

To construct. Draw the tangent AD. On AD describe a semicircle AGD, bisect it in G; with centre A and radius AG, describe a circle cutting the given circle in F. Join AF and produce it to meet the circumference again in C.

47. Let C be the centre of the given circle, AC the given radius; produce AC to meet the circumference in B. With centre B and radius equal to the given line, describe a circle cutting the given circle in D. Through D draw DE parallel to BA, meeting the circumference in E, at B make the angle CBG equal to the given angle, and from E, draw EF parallel to DB, meeting the given radius in F.

48. On any two sides of the triangle, describe segments of circles each containing an angle equal to two thirds of a right angle, the point of intersection of the arcs within the triangle will be the point required, such that three lines drawn from it to the angles of the triangle shall contain equal angles. Euc. III. 22.

49. Let AB, AC be the lines containing the given angle BAC, and let PQ be the line of given length. Bisect PQ in R and on PR describe a rectangle equal to the given area; also on PQ describe a segment of a circle containing an angle equal to the angle BAC, and let the arc of this segment cut the side of the rectangle opposite to PR in S. Join SP, SQ. On AB take AD equal to SP, and on AC take AE equal to SQ; join DE, then DAE is the triangle required. It may be shewn that the triangle cut off has the greatest possible area when it is isosceles.

50. Let A, B be the extremities of the diameter of the circle, CD the given chord not parallel to AB; and let KL be the length to be cut from CD by lines drawn from A, B, and meeting at a point in the circumference.

Draw BE equal to KL and parallel to CD, join AE and on AE describe a semicircle cutting CD in F, join AF and produce it to meet the circumference in H, and join HB cutting CD in G. Then FG is equal to KL, as may be shewn from BEFG being a parallelogram. If the semicircle touch CD there is only one solution, if it cut CD there are two, if it do not meet CD, there is no solution to the problem.

51. This problem is a slightly varied form of Problem 39, p. 317.

52. On any two sides AC, CB describe internally two segments of circles each containing an angle equal to one third of four right angles, and let the segments intersect each other in D. Join AD, BD, CD. Let a circle be described with centre A and radius AD and a tangent be drawn to it at D. Then the sum of BD, DC is a minimum when they make equal angles with the tangent, and therefore when they make equal angles with the radius AD. Similarly it may be shewn that the sum of AD, DC is a minimum. But the lines AD, BD, CD make equal angles with each other, D therefore is the point required.

53. Let $Aa$, $Bb$, $Cc$ be drawn from the angles to the bisections of the opposite sides of the triangle ABC. At B, $b$ draw BD, $b$D parallel to $a$A, $Cc$ respectively, and meeting each other in D. Join D$c$, $bc$. Then D$b$ is proved equal to C$c$ from the triangles D$b$A, $c$C$b$: also in a similar way DB is proved equal to $a$A.

54. Let ABC be the given isosceles triangle having the vertical angle at C, and let FG be any given line. Required to find a point P in FG such that the distance PA shall be double of PC. Divide AC in D so that AD is double of DC, produce AC

to E and make AE double of AC. On DE describe a circle cutting FG in P, then PA is double of PC. This is found by shewing that $AP^2 = 4 . PC^2$.

55. The equal sides of the equivalent isosceles triangle are each a mean proportional between the two sides of the scalene triangle. Euc. VI. 15.

56. This is made to depend upon constructing a right-angled triangle of which the perpendicular and the opposite angle are given.

57. Let ABC be the triangle required; BC the given base, BD the given difference of the sides, and BAC the given vertical angle. Join CD and draw AM perpendicular to CD. Then MAD is half the vertical angle and AMD a right angle; the angle BDC is therefore given, and hence D is a point in the arc of a given segment on BC. Also since BD is given, the point D is given, and therefore the sides BA, AC are given. Hence the synthesis.

58. On any base BC describe a segment of a circle BAC containing an angle equal to the given angle. From D the middle point of BC draw DA to make the given angle ADC with the base. Produce AD to E so that AE is equal to the given bisecting line, and through E draw FG parallel to BC. Join AB, AC and produce them to meet FG in F and G. The demonstration depends on Book VI.

59. Analysis. Let ABC be the triangle, and let the circle ABC be described about it; draw AF to bisect the vertical angle BAC and meet the circle in F, make AV equal to AC, and draw CV to meet the circle in T; join TB and TF, cutting AB in D; draw the diameter FS cutting BC in R, DR cutting AF in E; join AS, and draw AK, AH perpendicular to FS and BC. Then shew that AD is half the sum, and DB half the difference of the sides AB, AC. Next, that the point F in which AF meets the circumscribing circle is given, also the point E where DE meets AF is given. The points A, K, R, E are in a circle, Euc. III. 22. Hence KF . FR = AF . FE, a given rectangle; and the segment KR, which is equal to the perpendicular AH, being given, RF itself is given. Whence the construction.

60. On AB the given base describe a circle such that the segment AEB shall contain an angle equal to the given vertical angle of the triangle. Draw the diameter EMD cutting AB in M at right angles. At D in ED, make the angle EDC equal to half the given difference of the angles at the base, and let DC meet the circumference of the circle in C. Join CA, CB; ABC is the triangle required.

For, make CF equal to CB, and join FB cutting CD in G.

61. Let ABC be the triangle, AD the perpendicular on BC. With centre A, and AC the less side as radius, describe a circle cutting the base BC in E, and the longer side AB in G, and BA produced in F, and join AE, EG, FC. Then the angle GFC being half the given angle, BAC is given, and the angle BEG equal to GFC is also given. Likewise BE the difference of the segments of the base, and BG the difference of the sides, are given by the problem. Wherefore the triangle BEG is given (with two solutions). Again, the angle EGB being given, the angle AGE, and hence its equal AEG is given; and hence the vertex A is given, and likewise the line AE equal to AC the shorter side is given. Hence the construction.

62. On the given base AB describe a segment of a circle containing an angle equal to the given angle, and through C the bisection of the base draw the diameter DE perpendicular to AB. Join AE, and on AE describe a semicircle ACE. From A, place in this semicircle AF equal to half the given difference of the two sides of the triangle. Produce AF to meet the circle in G and join GB; ABG is the triangle required.

63. Let ABC be the triangle, D, E the bisections of the sides AC, AB. Join CE, BD intersecting in F. Bisect BD in G and join EG. Then EF, one third of EC is given, and BG one half of BD is also given. Now EG is parallel to AC; and the angle BAC being given, its equal opposite angle BEG is also given. Whence the segment of the circle containing the angle BEG is also given.

Hence F is a given point, and FE a given line, whence E is in the circumference of the given circle about F whose radius is FE. Wherefore E being in two given circles, it is itself their given intersection.

64. Of all triangles on the same base and having equal vertical angles, that triangle will be the greatest whose perpendicular from the vertex on the base is a maximum, and the greatest perpendicular is that which bisects the base. Whence the triangle is isosceles.

65. When the vertical angle and area are given, the rectangle under the sides is also given. Likewise the sum of the squares and the rectangle of the two lines which constitute the sides are given. These lines may hence be found, and the triangle constructed.

66. Let AB be the base, and AHB the segment containing the vertical angle. At any point H draw the tangent HP, making the square of HP equal to the given rectangle under the sum of the sides and one of them. Find O the centre of the circle AHB and join PO, with which as radius describe a circle PDQ. Also from R the middle of the arc AHB, with radius RA or RB describe the circle ADB cutting PDQ in D. Draw DA cutting AHB in C, and join CB. ACB is the triangle required. For, draw the tangent DE, and join HO, EO, DO. Then $HP^2 = DE^2 = AD \cdot DC = (AC + CB) \cdot CB$; for the triangle DCB is isosceles.

67. Let C be the centre of the given circle, B the given point in the circumference, and A the other given point through which the required circle is to be made to pass. Join CB, the centre of the circle is a point in CB produced. The centre itself may be found in three ways.

68. Let A be the given point, and B the given point in the given line CD. At B draw BE at right angles to CD, join AB and bisect it in F, and from F draw FE perpendicular to AB and meeting BE in E. E is the centre of the required circle.

69. Let AB, AC be the given lines and P the given point. Then if O be the centre of the required circle touching AB, AC in R, S, the line AO will bisect the given angle BAC. Let the tangent from P meet the circle in Q, and draw OQ, OS, OP, AP. Then there are given AP and the angle OAP. Also since OQP is a right angle, we have $OP^2 - QO^2 = OP^2 - OS^2 = PQ^2$ a given magnitude. Moreover the right-angled triangle AOS is given in species, or OS to OA is a given ratio. Whence in the triangle AOP there is given, the angle AOP, the side AP, and the excess of $OP^2$ above the square of a line having a given ratio to OA, to determine OA. Whence the construction is obvious.

70. Bisect the angle BAC by AF, at A draw AG perpendicular to AB and equal to half of DE. Through G draw GH parallel to AB, meeting AF in H. Then H is the centre of the required circle.

71. Draw any line AB so that AC is equal to BC; on this describe a segment to contain the given angle; through the given point P draw CQ, meeting the segment in Q, and join QA, QB; draw PD, PE parallel to QA, QB respectively; and DO perpendicular to AC, and EO to BC. Then O is the centre.

72. Let D be the given point and EF the given straight line. (fig. Euc. III. 32.) Draw DB to make the angle DBF equal to that contained in the alternate segment. Draw BA at right angles to EF, and DA at right angles to DB and meeting BA in A. Then AB is the diameter of the circle.

73. Let A, B be the given points, and CD the given line. From E the middle of the line AB, draw EM perpendicular to AB, meeting CD in M, and draw MA. In EM take any point F; draw FH to make the given angle with CD; and draw FG equal to FH, and meeting MA produced in G. Through A draw AP parallel to FG; and CPK parallel to FH. Then P is the centre, and C the third defining point of the circle required: and AP may be proved equal to CP by means of the triangles GMF, AMP; and HMF, CMP, Euc. VI. 2. Also CPK the diameter makes with CD the angle KCD equal to FHD, that is, to the given angle.

74. See Euc. III. 37, and Theo. 3, p. 313.

75. Let A be the base of the tower, AB its altitude, BC the height of the flagstaff, AD a horizontal line drawn from A. If a circle be described passing through the points, B, C, and touching the line AD in the point E: E will be the point required. Give the analysis.

76. Let A, B, C be the centres of the three given circles. On the same side draw the tangent DE to the circles whose centres are A, B; and FG to the circles whose centres are B, C; bisect DE in H, and FG in K; and draw HL, KM perpendicular to AB, AC, (the lines joining the centres of the circles) to meet in O. O is the point required. For, join DA, HA, FA, KA, EB, HB, KC, CG, AO, BO, CO; draw the tangents OP, OQ, OR to the circles whose centres are A, B, C respectively, and join AP, BQ, CR. Then the difference of the squares of OP and OQ may be shewn to be equal to the difference of the squares of DH and HE. But

DH is equal to HE by construction, it follows that OP is equal to OQ. In the same way it may be shewn, that OP is equal to OR.

NOTE. The line HO is called *the radical axis* of the circles whose centres are A and B; and similarly, KO of the circles whose centres are B and C.

The point O is *the radical centre* of the circles whose centres are A, B, C.

77. Let POQ be the common diameter, O being the point of contact of the circles B, C. Let DEOFG be any line drawn through O and meeting the circumferences of the circles. Join PE, QF, then DE is equal to FG by Theorem I. p. 344.

Can DE be shewn to be equal to FG without proportion?

78. Let the point E in AO be supposed to be found subject to the conditions of the Problem. Produce PO to meet the circumference in Q. Then by Euc. II. 5, 6, III. 36, Cor. combined with the given conditions, the squares of OE and OP may be shewn to be equal to five times the square of the radius AO. And the triangle PEO is right-angled. Hence the square of PE is five times the square of AO, and AO is known, therefore the line PE is known, also the point P is given. Whence the point E is determined subject to the given conditions.

79. This is the same as the following problem with only a slight alteration.

80. From A draw the chord AC equal to the side of the given square, join BC and produce it to meet AP in P, the point P thus found is the point required, as may be shewn by Euc. III. 36, I. 47, II. 3. The limits of the possibility may be readily determined.

81. On PC describe a semicircle cutting the given one in E, and draw EF perpendicular to AD: then F is the point required.

82. Let AB, AC be the two given lines meeting at the point A, and D the given point. From D draw DE perpendicular to AB and produce ED to F, so that the rectangle contained by ED, DF may be equal to the given rectangle. On DF as a diameter describe a circle cutting AC in G, join GE and produce it to meet AB in H. Then a circle may be described through the points E, G, F, H.

83. If the ladder be supposed to be raised in a vertical plane, the locus of the middle point may be shewn to be a quadrantal arc of which the radius is half the length of the ladder.

84. The point E will be found to be that point in BC from which two tangents to the circles described on AB and CD as diameters, are equal. Euc. III. 36.

85. Draw from the given point A a straight line AB, and divide AB on C, so that the rectangle AB, BC is equal to the given rectangle. If the line given by position (which is a circle) be described so as to pass through the points A, B, the locus of the extremities of all such lines drawn from A will be in the convex circumference of the circle.

86. Suppose AD, DB the tangents to the circle AEB and which contain the given angle. Draw DC to the centre C and join CA, CB. Then the triangles ACD, BCD are always equal: DC bisects the given angle at D and the angle ACB. The angles CAB, CBD, being right angles, are constant, and the angles ADC, BDC are constant, as also the angles ACD, BCD; also AC, CB the radii of the given circle. Hence the locus of D is a circle whose centre is C and radius CD.

87. (1) This locus may be shewn to be a circle, as the sum of the angles at the base is constant, the vertical angle is also constant. Euc. III. 21.

(2) This locus is a hyperbola, and was inserted here by an oversight.

88. The locus is a hyperbola. Also the equation to the locus may be put under the form, $4xy \tan\phi = (a^2 - b^2)\sec^2\phi$.

89. Join AB, and upon it describe a segment of a circle which shall contain an angle equal to the given angle. If the circle cut the given line, there will be two points; if it only touch the line, there will be one; and if it neither cut nor touch the line, the problem is impossible.

90. Describe a circle having the given line AB as its diameter and centre C. Draw any radius CD, and at D draw a tangent DE equal to AB and join CE. In AB produced, make CF equal to CE. F is the point required.

Let EC meet the circle in B', and when produced, in A'.

91. This is only a case of the preceding, instead of taking the tangent DE equal to AB, it must be taken equal to the side of the given square.

92. If the rectangle of the sides of the right-angled triangle be equal to the

square of their difference, it may be shewn that the rectangle is also equal to one third of the square of the hypothenuse. Hence, if a semicircle be described on AB the given hypothenuse, and a line AC be drawn at right angles to AB and equal to one third of AB, and from C a line CD be drawn parallel to AB meeting the semicircle in D, and DA, DB be joined: DAB is the required triangle.

93. Let ABC be the required triangle in which are given AD, BE, CF, lines drawn from the angles to the bisections of the opposite sides of the triangle, and intersecting one another in G. Then AG is double of GD. Produce GD to H, making DH equal to DG and join BH, CH. Then BGCH is a parallelogram, and the sides of the triangle BHG are respectively each two thirds of the lines drawn from the angles to the bisections of the sides.

94. Let AB, BC, CA be the three given lines, and D the given point. Draw any line DE through D to meet AC in E; and any line FG parallel to DE to meet AB in F, and BC in G; and join EF, EG; and lastly, through F and G, draw parallels to EG and EF meeting DE in R and P, and each other in Q. Then the triangle PQR fulfils the conditions.

It is obvious that there is no limit to the number of triangles which can be constructed to fulfil the given conditions. For the *direction* of DE from D is arbitrary; and likewise the distance of the parallels DE, FG.

95. Let ABCD be the given square, and let the diagonals intersect each other in E. Let F be any point in the locus, and join FA, FB, FD, FC, FE. Then by Theorem 22, p. 309, the sum of the squares of the lines from F to the angles of the square are shewn to be equal to twice the area of the square and four times the square of EF. And E is a fixed point; if EF be constant, the locus of F is a circle whose centre is E and radius EF.

## HINTS, &c. TO THE THEOREMS. BOOK III.

5. First. Join the extremities of the chords, then Euc. I. 27; III. 28.

Secondly. Draw any straight line intersecting the two parallel chords and meeting the circumference.

6. This is the converse of the former part of the preceding Theorem 5.

7. Let the circles intersect in A, B; and let CAD, EBF be any parallels passing through A, B and intercepted by the circles. Join CE, AB, DF. Then the figure CEFD may be proved to be a parallelogram. Whence CAD is equal to EBF.

8. Construct the figure and the arc BC may be proved equal to the arc B′C′.

9. See Theorem 67, p. 351.

10. Let AB be drawn from the given point A, touching the given circle whose centre is D. Join AD, DB, then the position of the point B may be shewn to be given, and therefore the line AB both in position and magnitude. (Euclid's Data, Prop. 94.)

11. Let C be the centre of the circle, and E the point of contact of DF with the circle. Join DC, CE, CF.

12. Let AB, AC be the sides of a triangle ABC. From A draw the perpendicular AD on the opposite side, or opposite side produced. The semicircles described on AB, BC both pass through D. Euc. III. 31.

13. Let A be the right angle of the triangle ABC, the first property follows from the preceding Theorem 12. Let DE, DF be drawn to E, F the centres of the circles on AB, AC, and join EF. Then ED may be proved to be perpendicular to the radius DF of the circle on AC at the point D.

14. Let ABC be a triangle, and let the arcs be described on the sides externally containing angles, whose sum is equal to two right angles. It is obvious that the sum of the angles in the remaining segments is equal to four right angles. These arcs may be shewn to intersect each other in one point D. Let $a$, $b$, $c$ be the centres of the circles on BC, AC, AB. Join $ab$, $bc$, $ca$; A$b$, $b$C, C$a$, $a$B; B$c$, $c$A; $b$D, $c$D, $a$D. Then the angle $cba$ may be proved equal to one half of the angle A$b$C. Similarly, the other two angles of $abc$.

15. The angle OAQ may be proved equal to the angle OCQ, and the truth of the theorem is manifest from Euc. I. 32.

16. Bisect the lines and join the points of bisection with the centre of the circle; the two triangles thus formed may be shewn to be equal, and by Euc. III. 14, the equality of the lines is inferred.

17. Let the diagram be drawn, and from the centre of the circle draw a perpendicular on the chord which passes through the middle points of the two equal chords. Then Euc. III. 3.

18. Let AB a chord in a circle be bisected in C, and DE, FG two chords drawn through C; also let their extremities DG, FE be joined intersecting CB in H, and AC in K; then AK is equal to HB. Through H draw MHL parallel to EF meeting FG in M, and DE produced in L. Then by means of the equiangular triangles, HC may be proved to be equal to CK, and hence AK is equal to HB.

19. Let AB, AC be the bounding radii, and D any point in the arc BC. The circle described on AD will always be of the same magnitude, and the angle EAF in it, is constant:—whence the arc EDF is constant, and therefore its chord EF.

20. This is manifest from Euc. III. 23.

21. Join BC. Since the angles at B, C are right angles, a circle may be described about the figure BECF. Euc. III. 22. Let the circle be described. Then the angle BEF is equal to BCF or BCA; and BCA is equal to BDA or FDG; also BFE is equal to DFG: whence two of the angles of the triangle FGD being equal respectively to two angles of the triangle BFE; the third angle FGD is equal to the third angle EBF, which is a right angle, Euc. III. 31.

22. The chord PQ is proved greater than any other chord TR passing through the same point N, by Euc. I. 19; III. 15. If a circle be described about the triangle PSQ, it will touch the circumference of the given circle in P, and the angle SPQ may be shewn to be greater than the angle STR.

23. The perpendicular from the vertex bisects the base of the isosceles triangle, and the circle described upon one of the sides will pass through the bisection of the base, Euc. III. 31.

24. Let AB, CD be any two diameters of a circle, and let two other circles through B, D cut the diameters in E, F; and in G, H. Join BD. Then DE, BF may be proved to be equal, as also BD, DG.

25. Constructing the figure and producing the tangent QP, the triangle CPQ may be shewn to be an isosceles triangle, as also the triangle C'QP: also CQ (not CQ') and C'Q may be each shewn to be equal to QP.

26. The chord AB is common to the two equal circles. The angles ADB, ACB may be shewn to be equal. Hence the triangle BCD is isosceles.

27. This is only the extreme case of Theorem 5, p. 320. Also the angle contained by the tangents may be shewn to be equal to the difference between the angles in the two segments formed by joining the points of contact.

28. By constructing the figure and joining BC, the truth will appear from Euc. I. 32; and III. 20.

29. It may be remarked, that generally, the mode by which, in pure geometry, three lines must, under specified conditions, pass through the same point, is that by reductio ad absurdum. This will for the most part require the converse theorem to be first proved or taken for granted.

The converse theorem in this instance is, "If two perpendiculars drawn from two angles of a triangle upon the opposite sides, intersect in a point, the line drawn from the third angle through this point will be perpendicular to the third side." Theorem 21, supra, is the same under a modified form of expression.

The proof will be formally thus, taking the same figure as in Theorem 21 supra. Let EHD be the triangle, AC, BD two perpendiculars intersecting in F. If the third perpendicular EG do not pass through F, let it take some other position as EH; and through F draw EFG to meet AD in G. Then it has been proved that EG is perpendicular to AD: whence the two angles EHG, EGH of the triangle EGH are equal to two right angles:—which is absurd.

30. The truth of this appears at once from Euc. III. 21.

31. Since all the triangles are on the same base and have equal vertical angles, these angles are in the same segment of a given circle. The lines bisecting the vertical angles may be shewn to pass through the extremity of that diameter which bisects the base.

32. This is the converse of the preceding Theorem.

33. It can be shewn that of all triangles on the same base and between the same parallels, the isosceles triangle has the least perimeter. The equilateral triangle, being also isosceles, may be shewn to be greater than any other isosceles triangle of the same perimeter, and hence of all triangles with equal perimeters, the equilateral has the greatest area.

34. Apply Euc. III. 31.

35. Let AC be the common base of the triangles, ABC the isosceles triangle, and ADC any other triangle on the same base AC and between the same parallels AC, BD. Describe a circle about ABC, and let it cut AD in E and join EC. Then, Euc. I. 17, III. 21.

36. Let two lines AP, BP be drawn from the given points A, B, making equal angles with the tangent to the circle at the point of contact P, take any other point Q in the convex circumference, and join QA, QB: then by Prob. 1, p. 293, and Euc. I. 21.

37. Let DKE, DBO (fig. Euc. III. 8) be two lines equally inclined to DA, then KE may be proved to be equal to BO, and the segments cut off by equal straight lines in the same circle, as well as in equal circles are equal to one another.

38. Produce the radii to meet the circumference. See Theorem 27, p. 321.

39. Let F be any point in the diameter AD of a circle whose centre is E (fig. Euc. III. 7) and let HFK, KFL, LFM, &c. be equal angles at F, then the arc HK is less than KL, KL less than LM, &c. Join HK, KL, LM, &c. and prove HK less than KL. Take FN equal to FH, and join KN; KN is equal to KH. Produce KF to meet the circumference in P, and join LP, HP, LH. Then the angle KNL may be proved to be greater than KLN.

40. Join the point of intersection with the centre of the circle and let fall from the centre perpendiculars upon the chords.

41. The diagram of Euc. III. 7, suggests the method of proof.

42. See Theorem 16, p. 321.

43. If BE intersect DF in K (fig. Euc. III. 37). Join FB, FE, then by means of the triangles, BE is shewn to be bisected in K at right angles.

44. The angle between the chord BE and the diameter BFM may be shewn equal to the angle between the tangent BD and the line DF drawn from D through the centre of the circle. (fig. Euc. III. 37.)

45. Let AB, CD be any two diameters of a circle, O the centre, and let the tangents at their extremities form the quadrilateral figure EFGH. Join EO, OF, then EO and OF may be proved to be in the same straight line, and similarly HO, OK.

Note.—This Proposition is equally true if AB, CD be any two chords whatever. It then becomes equivalent to the following proposition:—The diagonals of the circumscribed and inscribed quadrilaterals, intersect in the same point, the points of contact of the former being the angles of the latter figure.

46. Let the chord AB, of which P is its middle point, be produced both ways to C, D, so that AC is equal to BD. From C, D, draw the tangents to the circle forming the tangential quadrilateral CKDR, the points of contact of the sides, being E, H, F, G. Let O be the centre of the circle. Join EH, GF, CO, GO, FO, DO.

Then EH and GF may be proved each parallel to CD, they are therefore parallel to one another. Whence is proved that both EF and DG bisect AB.

47. Let ABC be the *isosceles* triangle which fulfils the conditions; take any point F in the base BC, from which draw FH and FG parallel to AB and AC; make GD equal to GA and draw DFE. Then since the side AG of the triangle ADE is bisected in G, and GF is parallel to AE, the base DE is bisected in F. Again, since GF is equal to AH, and FH to AG; and since BGF, HFC, are isosceles triangles; it follows that $AG + AH = AG + GB = AB$. Wherefore $AD + AE = 2 . AG + 2 . AH = 2 . AB = BA + AC$. The variable triangle ADE therefore has its vertical angle and the sum of its sides constant, and the middle of its base is in the line BC.

48. Let the tangent AB touch the circle in C, and let CD be drawn from C perpendicular on any diameter EF, and let the perpendiculars from E and F meet the tangent in B and A respectively. Join CE, CF. Then the angles BCE, DCE, may be shewn each equal to the angle CFE; and by means of the triangles BCE,

DCE, BE may be shewn equal to ED, and in a similar way FA may be shewn equal to FD.

49. The line drawn from the point of intersection of the two lines to the centre of the given circle may be shewn to be constant, and the centre of the given circle is a fixed point.

50. Let AD, DF be two lines at right angles to each other; O, the centre of the circle BFQ; A, any point in AD from which tangents AB, AC are drawn; then the chord BC shall always cut FD in the same point P, wherever the point A is taken in AD. Join AP. Then BAC is an isosceles triangle: and

$$FD.DE+AD^2=AB^2=BP.PC+AP^2=BP.PC+AD^2+DP^2.$$

$$\text{Or again, } BP.PC=FD.DE-DP^2.$$

The point P, therefore, is independent of the position of the point A; and is consequently the same for all positions of A in the line AD.

51. Let C be the point without the circle from which the tangents CA, CB are drawn, and let DE be any diameter, also let AE, BD be joined, intersecting in P, then if CP be joined and produced to meet DE in G: CG is perpendicular to DE. Join DA, EB and produce them to meet in F.

Then the angles DAE, EBD being angles in a semicircle, are right angles; or DB, EA are drawn perpendicular to the sides of the triangle DEF: whence GPCF is perpendicular to the third side DE. See Theorem 29, p. 321.

52. Let AB, AC be drawn from A and touch the circle in B, C; let AB be perpendicular to the diameter BD, and CE perpendicular to BD, also let AD intersect CE in F, then CE is bisected in F. Join DC and produce it to meet BA produced in G. DG may be shewn to be equal to AD, and EF to FB by means of similar triangles. Euc. VI. 4.

53. This is the same as Theorem 44.

54. Let the radius BC produced meet the circumference of the quadrantal arc when continued in F, and join FE, CD, BE. Then FE is parallel to CD, and the angles DEB, EBD may be each shewn to be equal to half a right angle.

Each of the tangents to the larger circle at A and B makes with AB an angle equal to half a right angle, it follows that if AD make with AB an angle *less* than half a right angle, AD must cut the arc of the quadrant.

55. By constructing the diagram in accordance with the directions given, it will be found that two circles described from the centres B, H, and with radius BH, *do not intersect each other* in the centre of the circle ABF: it would hence appear that there is some inaccuracy in the terms or letters of the enunciation.

56. Let AB be the diameter of the given circle of which the centre is C, and E the bisection of any chord AD. Join EC, then the angle AEC may be proved to be always a right angle in whatever position the chord AD may be situated.

57. Join AD, and the first equality follows directly from Euc. III. 20, I. 32. Also by joining AC, the second equality may be proved in a similar way. If however the line AD do not fall on the same side of the centre O as E, it will be found that the *difference*, not the *sum* of the two angles, is equal to 2.AED. See note to Euc. III. 20, p. 108.

58. This problem cannot be constructed by the line and circle, as the Algebraical equation which arises for finding D is of the third degree.

59. See Theorem 85, p. 325.

60. Complete the circle whose segment is ADB; AHB being the other part. Then since the angle ACB is constant, being in a given segment, the sum of the arcs DE and AHB is constant. But AHB is given, hence ED is also given and therefore constant.

61. Let A, B, be the centres, EF the tangent intersecting the line AB joining the centres, and CD the other tangent. Draw the radii AC, AE, BD, BF to the points of contact; and draw BG parallel to DC meeting AC in G; and BH parallel to FE meeting AE produced in H. Then $BG=CD$, $BH=EF$, $AG=AC-BD$, and $AH=AE+BF=AC+BD$: and $CD^2-EF^2$ may be proved to be equal to $4.AC.BD$, or the rectangle of the diameters of the circles.

62. Let the segments AHB, AKC be externally described on the given lines AB, AC, to contain angles equal to BAC. Then by the converse to Euc. III. 32, AB touches the circle AKC, and AC the circle AHB.

63. See Theorem 82, p. 325.

64. Let A, B, be the centres, and C the point of contact of the two circles; D, E the points of contact of the circles with the common tangent DE, and CF a tangent common to the two circles at C, meeting DF in E. Join DC, CE. Then DF, FC, FE may be shewn to be equal, and FC to be at right angles to AB.

65. The possibility is obvious, and the centre of the required circle will be found to be the point of intersection of two circles described from the centres of the given circles with their radii increased by the radius of the required circle.

66. This may be directly shewn from Euc. III. 36, 37.

67. See Theorem 67, p. 351.

68. This is the same as Theorem 26, p. 321: repeated by mistake.

69. This follows directly from Euc. III. 36.

70. The line drawn through the point of contact of the two circles parallel to the line which joins their centres, may be shewn to be double of the line which joins the centres, and greater than any other straight line drawn through the same point and terminated by the circumferences. The greatest line is therefore dependent on the distances between the centres of the two circles.

71. A repetition of Theorem 7, p. 320.

72. This is the same as Theorem 60, p. 324, under another form of expression.

73. This is at once obvious from Euc. III. 36.

74. Each of the lines CE, DF may be proved parallel to the common chord AB.

75. This may be proved by shewing that the line joining the centres, bisects the angles at the centres which are contained by radii drawn to the points of intersection of the circles.

76. By constructing the figure and applying Euc. I. 8, 4, the truth is manifest.

If two diameters from one of the points of intersection be drawn in both circles, and the other extremities of them be joined with the other intersection of the circles: then these two lines are in the same straight line.

77. The third circle must be defined as that whose radius is equal to the diameter of either of the equal circles; or else that *touch* must be used instead of *cut*, in the enunciation. For in that case only will the angles at the other point of intersection be right angles.

78. Let E be that point in the circumference of one circle which is the centre of the other. Join CE and produce it to meet the circumference in F. Join also FA, EA, OA. The triangles FEA, OHA may be proved to be equiangular.

79. Let the two circles touch one another in the point C, and let A, B be any two points in the circumference of the interior circle, and D any point in the circumference of the exterior circle; the angle ACB is greater than the angle ADB. Let AD intersect the interior circumference in E and join EB.

80. Let perpendiculars from the centre of the larger circle be drawn on those straight lines, then Euc. III. 15.

81. Let the line which joins the centres of the two circles be produced to meet the circumferences, and let the extremities of this line and any other line from the point of contact be joined. From the centre of the larger circle draw perpendiculars on the sides of the right-angled triangle inscribed within it. In the enunciation of the Theorem, for *externally* read *internally*.

82. This is only a slight variation of Theorem 63, p. 324.

83. The sum of the distances of the centre of the third circle from the centres of the two given circles, is equal to the sum of the radii of the given circles, which is constant.

84. This may be shewn to follow directly from Euc. III. 36, and I. 47.

85. Let the circles touch at C either externally or internally, and their diameters AC, BC through the point of contact will either coincide or be in the same straight line. DCE any line through C will cut off similar segments from the two circles. For joining AD, BE, the angles in the segments DAC, EBC are proved to be equal.

The remaining segments are also similar, since they contain angles which are supplementary to the angles DAC, EBC.

86. In the enunciation, read "are *not* similar" instead of "are similar." The lines joining the common centre and the extremities of the chords of the circles, may be shewn to contain unequal angles, and the angles at the centres of the circles are dou-

ble the angles at the circumferences, it follows that the segments containing these unequal angles are not similar.

87. Let AB, AC be the straight lines drawn from A, a point in the outer circle to touch the inner circle in the points D, E, and meet the outer circle again at B, C. Join BC, DE. Prove BC double of DE.

Let O be the centre, and draw the common diameter AOG intersecting BC in F, and join EF. Then the figure DBFE may be proved to be a parallelogram.

88. Let A, B, C, be the centres of the three equal circles, and let them intersect each other in the point D: and let the circles whose centres are A, B intersect each other again in E; the circles whose centres are B, C in F; and the circles whose centres are C, A in G. Then FG is perpendicular to DE; DG to FC; and DF to GE. Since the circles are equal, and all pass through the same point D, the centres A, B, C are in a circle about D whose radius is the same as the radius of the given circles. Join AB, BC, CA; then these will be perpendicular to the chords DE, DF, DG. Again, the figures DAGC, DBFC, are equilateral, and hence FG is parallel to AB; that is, perpendicular to DE. Similarly for the other two cases.

89. This is true not only for three circles, but for all circles.

Let A be the point of contact, C, the centre of any circle in the line AP, and B the given point from which the tangents are to be drawn. Join BA and make the angle ABF equal to BAF, and produce BF till FG be equal to FB. The chord of contact DE will always pass through G. For join BC cutting DE in H: then BHG is a right angle. Also BF, FA, FG are equal to one another, and A is in a semicircle on BG, and AG being joined, BAG is a right angle. The angles BHG, BAG, therefore are right angles in the same semicircle; and hence HE always passes through G, the extremity of the hypothenuse of the triangle BAG.

90. Let E be the centre of the circle which touches the two equal circles whose centres are A, B. Join AE, BE which pass through the points of contact F, G. Whence AE is equal to EB. Also CD the common chord bisects AB at right angles, and therefore the perpendicular from E on AB coincides with CD.

91. Let the three chords be AB, AC, AD, the middle points of which are $b$, $c$, $d$; let E, F, G be the intersections of the circles on AB and AC, on AB and AD, and on AC and AD respectively; join BC, CD, DB, $bc$, $cd$, $db$, AE, AF, AG; and let $m$ be the intersection of $bc$, AE. The proof may be made to depend on the following theorem;—If a circle be inscribed in a triangle and perpendiculars be drawn upon the sides from any point in the circumference; the three points of intersection are in the same straight line.

92. By joining the points of intersection of the circles, the foursided figure so formed may be shewn to have its opposite sides equal and its angles right angles. The diagonals may easily be shewn to cut one another at the centre of the middle circle.

93. Let three circles touch each other at the point A, and from A, let a line ABCD be drawn cutting the circumferences in B, C, D. Let O, O′, O″ be the centres of the circles, join BO, CO′, DO″, these lines are parallel to one another. Euc. I. 5, 28.

94. Let the fixed circle CDE be cut in C, D by any circle whatever passing through the fixed points A, B: draw CD to meet BA produced in F. Then BF.FA = DF.FC; and hence F is independent of the magnitude of the circle ACDB; and is consequently the same for all, that is, all the chords pass through the same point F.

95. This is similar to the last, except that the rectangle AF.FB is exchanged for the square of the tangent.

96. Let any number of circles touch each other internally at the point A, and let a common tangent be drawn at A. With any point B in the tangent and any radius, describe a circle cutting the given circles in C, D, E, &c. Join BC, BD, BE, &c., and produce them to meet the circles again in C′, D′, E′, &c. Then CC′, DD′, EE′, &c., may be shewn to be equal to one another by Euc. III. 36.

97. The construction of the figure suggests a reference to Theorem 22, p. 309.

98. Let AB be a chord parallel to the diameter FG of the circle, fig. Theo. 1, p. 235, and H any point in the diameter. Let HA and HB be joined. Bisect FG in O, draw OL perpendicular to FG cutting AB in K, and join HK, HL, OA. Then the square of HA and HF may be proved equal to the squares of FH, HG by Theo. 20, p. 233; Euc. I. 47; Euc. II. 9.

99. Let the chords AB, CD intersect each other in E at right angles. Find F the centre, and draw the diameters HEFG, AFK and join AC, CK, BD. Then by Euc. II. 4, 5; III. 35.

100. Let ABCD be any quadrilateral figure, AC, BD the diagonals, F, G, their points of bisection, E the point of bisection of FG. Let P be any point in the circumference of a circle described from centre E. Join PF, PE, PG, PA, PB, PC, PD, EA, EB, EC, ED. Then by Theorem 22, p. 309.

101. Let the figure be constructed and join DB, DG. Then DABG is a rectangle, and FG is equal to CD, both being diameters of the circle, and by Euc. I. 47; II. 12, 3; III. 36, the property may be proved.

In the enunciation read, "or their lines produced in H, G."

102. This is only another form of stating Theorem, 103.

103. This is manifest from Euc. III. 35.

104. Let ACB be the given acute angle at C the centre of the circle, which is subtended by the arc AB, and suppose ACD to be one third of the angle ACB. Through B draw BEF parallel to CD and meeting the circumference in E and the radius AC produced in F. Join CE, then FE may be shewn equal to the radius CE. If therefore from B the line BEF, &c.

105. Let Q be the centre of the circles. Join QO, QB. Then QOB is a right-angled triangle. Also ON may be proved equal to OM by Theorem 18, p. 321. And $4 \,.\, CN \,.\, NF + MO^2$ may be shewn to be equal to the difference of the squares of the radii of the circles by Euc. III. 35; II. 6; I. 47.

106. Let the figure be constructed, and the truth is obvious from Euc. I. 41.

107. Let E, F be the points in the diameter AB equidistant from the centre O; CED any chord; draw OG perpendicular to CED, and join FG, OC. The sum of the squares of DF and FC may be shewn to be equal to twice the square of FE and the rectangle contained by AE, EB, by Euc. I. 47; II. 5; III. 35.

108. Let the chords AB, AC be drawn from the point A, and let a chord FG parallel to the tangent at A be drawn intersecting the chords AB, AC in D and E, and join BC. Then the opposite angles of the quadrilateral BDEC are equal to two right angles, and a circle would circumscribe the figure. Hence by Euc. I. 36.

109. Let the line drawn from A touch the circumference in P, and from B, C, let lines be drawn to the points where the circle intersects the two sides of the triangle. Then by Euc. III. 31; II. 13; III. 36.

110. Let QOP cut the diameter AB in O. From C the centre draw CH perpendicular to QP. Then CH is equal to OH, and by Euc. II. 9, the squares of PO, OQ are readily shewn to be equal to twice the square of CP.

111. From P draw PQ perpendicular on AB meeting it in Q. Join AC, CD, DB. Then circles would circumscribe the quadrilaterals ACPQ and BDPQ, and then by Euc. III. 36.

112. In the enunciation for "centre" read "circumference;" and for $KG^2$ read $AH^2$. Draw AGDK, AE, AC, AH, EK, KC, CG, CE, and let M be the intersection of AK, EC. Then AEKC is a rhombus, and $MK = MA$: also $GM = MD$. Whence $GK = AD = 2 \,.\, AG$, and $AK = 3 \,.\, AG$; wherefore $AK \,.\, AG = 3 \,.\, AG^2$. Also $AH^2 = AC^2 = AG^2 + GC^2 + 2 \,.\, AG \,.\, GM = 3 \,.\, GM^2$. Hence $AK \,.\, AG = AH^2$.

113. Here A is the extremity of the diameter and C the centre of the larger circle, the perpendicular BDE meets AC in E, the smaller circle in D and the larger in B. From the right-angled triangles the truth of the property may be shewn.

114. Describe the figure according to the enunciation; draw AE the diameter of the circle, and let P be the intersection of the diagonals of the parallelogram. Draw EB, EP, EC, EF, EG, EH. Since AE is a diameter of the circle, the angles at F, G, H are right angles, and EF, EG, EH are perpendiculars from the vertex upon the bases of the triangles EAB, EAC, EAP. Whence by Euc. II. 13, and Theorem 22, page 309, the truth of the property may be shewn.

115. Let AC, BC be produced to meet the circumference in A′, B′: produce also $QRr$ to meet the circumference in $q$. Through $r$ draw a line perpendicular to $Qq$ and meeting the circumference in S, $s$, and join $Sq$, then $Srq$ may be shewn to be a right-angled triangle having the sides $Sq$, $Sr$, $qr$ respectively equal to AB, $Qr$, QR.

116. If AQ, AP′ be produced to meet, these lines with AA′ form a right-angled triangle.

117. This Theorem is the same as 119, p. 361.

118. Let the tangents TP, TQ be drawn from any point T in the perpendicular CT to meet the circles in P, Q respectively. Join AP, PQ, then by Euc. I. 47. The square of PT may be shewn to be equal to the square of QT.

119. This theorem requires the aid of proportion. The locus of the point D which fulfils the condition is a circle (Theorem 17, p. 353.) whose diameter is found by drawing the common tangent PQ to meet ACB produced in K. Let AB meet the circles in L and M, and DC meet them in E and F; and join FL, EO, DK. Then since EO, FL are parallel to DK the base of the triangle CDK, and by compounding the two proportions deduced from Euc. VI. 2: we have

$$ED.DF : LK.KM :: CD^2 : CK^2,$$

and it may be shewn that $LK.KM = CK^2$;

$$\text{hence, } ED.DF = DC^2, \text{ or, } ED : DC :: DC : DF;$$
$$\text{whence, } ED.DC : DC^2 :: DC^2 : DC.DF;$$
$$\text{or, } DH^2 : DC^2 :: DC^2 : DG^2;$$
$$\text{and } DH : DC :: DC : DG; \text{ or } DH.DG = DC^2.$$

120. Let the figure be constructed, and let the perpendicular from AG on the diameter be greater than the perpendicular BH. Take O the centre of the circle, join CO, and draw BK perpendicular to AG. Then the triangles ABK, OCF being equiangular; AB is to BK or GH as OC is to CF. But DE is equal to twice OC, and CT is twice CF. Hence AB is to GH as DE is to CF; and therefore the rectangle contained by AB, CF is equal to that contained by GH, DE. This Theorem is misplaced, as it involves proportion.

121. Let AD meet the circle in G, H, and join BG, GC. Then BGC is a right-angled triangle and GD is perpendicular to the hypothenuse, and the rectangles may be each shewn to be equal to the square of BG. Euc. III. 35; II. 5; I. 47. Or, if EC be joined, the quadrilateral figure ADCE may be circumscribed by a circle. Euc. III. 31, 22, 36, Cor.

122. Join EC, ED, FG, FH, then by means of the similar triangles, two proportions may be found from which it is shewn, that the first rectangle is equal to the second, and the second to the third.

123. Let ADBC be the inscribed quadrilateral; let AC, BD produced meet in O, and AB, CD produced meet in P, also let the tangents from O, P meet the circles in K, H respectively. Join OP, and about the triangle PAC describe a circle cutting PO in G and join AG. Then A, B, G, O may be shewn to be points in the circumference of a circle. Whence the sum of the squares of OH and PK may be found by Euc. III. 36, and shewn to be equal to the square of OP.

124. This Theorem is only a different form of stating Theorem 119, p. 361. The consideration of it may be deferred, as it properly involves the idea of Harmonic proportionals.

125. This is an extension of Theorem 4, p. 314, where the chords intersect each other outside the circle. The truth of it may be readily shewn by drawing perpendiculars on the chords from the centre; and by Euc. II. 9; I. 47.

126. This Theorem involves the property:—that the circumferences of circles are proportional to their radii or diameters. In general, the locus of a point in the circumference of a circle which rolls within the circumference of another, is a curve called the *Hypocycloid;* but to this there is one exception, in which the radius of one of the circles is double that of the other: in this case, the locus is a straight line, as may be easily shewn from the figure.

127. Let the diagram be described as in the enunciation, and let CS meet the circle at A (between C and S); draw the tangent at A, and in it take $AE = AE'$ $= AS$; on CD describe a circle cutting the given one in B, B'; join BB' meeting CS in D; draw DE, DE'. Then these lines will be the loci of the point Q.

Analysis. Suppose it true that $QM = Sy$, Q being taken in the line DE. Then draw CP and produce it to meet the circle on SC in N, and join NS.

Since $Sy$ and NP are perpendicular to $Py$, they are parallel, and since CNS is in a semicircle, it is a right angle. Whence $Ny$ is a rectangle, and $NP = Sy$.

Again, by the similar triangles EAD, QMD, we have

$$QM.AD = EA.MD, \text{ or } QM(AC - CD) = (CS - CA)(CM - CD).$$

Also by the similar triangles CMP, CNS, we have

$CM . CS = CP . CN = CP^2 + CP . PN = CA^2 + CA . Sy$, or $CA . Sy = CM . CS - CA^2$.

Now since the proposition is assumed to be true, namely, that $QM = Sy$, a comparison of their values gives

$$CA : CA - CD :: CM . CS - CA^2 : (CS - CA)(CM - CD),$$

or $CD : CA :: CM . CS - CA^2 - CS . CM + CS . CD + CA . CM - CA . CD : CM . CS - CA^2$;

and since $CD : CA :: CA : CS$, and $CS . CD = CA^2$;

this becomes $CA : CS :: CA . DM : CM . CS - CA^2$;

Whence $CS . DM = CM . CS - CA^2$, or $CA^2 = CS (CM - DM) = CS . SD$, a known truth. Whence the Theorem is true.

128. Let PAB, PDC be the straight lines from P cutting the circle in A, B, C, D. Join PO, and about the triangle PAC describe a circle cutting PO in G, and join AG. Draw the diameter through P, and the centre Q, and through O a perpendicular to PQ cutting it in R, and the circle in H, K; draw HP, KP. Then it may be shewn that the points A, B, G, O, are in the circumference of a circle, also HK is bisected in R. Next by Euc. III. 36, 34, 37; II. 5; I. 47, $HP^2$ may be proved to be equal to $CP . PD$; or HP is a tangent drawn from P to the circle. In the same way, PK is a tangent. Whence O is situated in the chord which joins the points of contact of tangents drawn from P to the circle.

129. First. Since the angle PCQ and base PQ of the triangle PCQ are constant, the circle about PCQ is constant in magnitude, and consequently in diameter. Also since the angles PCQ, PRQ are supplementary, R is in the circumference of the circle PCQ. But RQC, RPC being right angles, CR is a diameter; and it has been proved to be of constant magnitude. Wherefore R is always at the same distance from C. Secondly. Draw RS, SC; then PQ is bisected in L, since PSQR is a parallelogram. Also bisect RC in K and join KL. Then since RC is the diameter of a circle given in magnitude, and PQ a given chord in it, the line KL is of constant magnitude. Moreover, since RS, RC are bisected in L and K, CS is equal to twice LK a given line; and the locus of S is a circle.

130. The locus may be shewn to be the circumference of the circle described on the base of the triangle as a diameter.

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## HINTS, &c. TO THE PROBLEMS. BOOK IV.

4. Draw through the centre a diameter parallel to the given line, and from its extremities, draw two lines perpendicular on the diameter.

5. This is a more general form of Problem 50, p. 217.

6. Place in the circle a straight line AB equal to the given line. Through the centre O draw a line OC perpendicular to AB, and with centre O and radius OC, describe a circle, and through P the given point, draw a straight line touching this circle. Hence, chords in one circle, which are also tangents to another concentric circle, are bisected at the points of contact.

7. Trisect the circumference and join the centre with the points of trisection.

8. Let AB, CD be two diameters intersecting each other in O at right angles, and suppose EFGH the straight line required which is trisected in the points F, G. At F in AF make the angle AFK equal to the angle AFE and join KH; then KF is equal to EF, KFH is a right-angled triangle of which the base KH is the chord of a quadrant. The semicircle described on KH as a diameter touches the diameter AB of the given circle in F, also the point O is given. Hence the construction.

9. If three lines be drawn within the triangle from the angular points, making equal angles with the respective sides of the triangle, an equilateral triangle is formed by the intersection of these lines, except when each line makes equal angles with *two* sides of the triangle: and the smaller the angles are which are contained between the lines and the sides of the triangle, the greater will be the area of the new triangle, which cannot exceed the area of the given triangle. If however the lines are not

required to be drawn within the triangle, the greatest triangle will be that whose sides are parallel to the sides of the given triangle.

10. This may be effected by Euc. IV. 10, 3.

11. From the vertex of the isosceles triangle let fall a perpendicular on the base. Then, in each of the triangles so formed, inscribe a circle, Euc. IV. 4; next inscribe a circle so as to touch the two circles and the two equal sides of the triangle. This gives one solution: the problem is indeterminate.

12. The meaning of this problem does not appear very obvious.

13. Let AB be the base of the given segment, C its middle point. Let DCE be the required triangle. From C draw CF perpendicular to the base DE, and make CH equal to the given line. Join HD and produce it to meet AB produced in K. Then FK is double of DF, and CH double of CF. Draw DL perpendicular to CK.

14. The first part is another form of stating Euc. IV. 5. In order that a circle may pass through four points, the condition may be deduced from Euc. III. 22, 35 or 36, Cor.

15. Apply Euc. IV. 5.

16. Let P, Q, R, be the three given points. Draw PR and take PR.PD = PA.PB; draw QD, and take QD.QE = QB.QC; through E, apply the chord CF to subtend the angle PDQ: C is an angular point in the circumference.

If the three points are in the same straight line, take PQ.PG = PA.PB, and RG.GH = GA.GV: draw the tangent HV, then VC, parallel to PQ, determines C an angular point. Swale's *Apollonius*, p. 48.

17. From the given angle draw a line through the centre of the circle, and at the point where the line intersects the circumference, draw a tangent to the circle, meeting two sides of the triangle. The circle inscribed within this triangle will be the circle required.

18. If four points successively be taken in the sides at equal distances from the angles, the lines joining these points will form a square. When the four points coincide with the bisections of the sides of the given square, the area of the inscribed square is a minimum.

19. Let the diagonals of the rhombus be drawn; the centre of the inscribed circle may be shewn to be the point of their intersection.

20. On the diameter AB describe a rectangle equal to the given rectilineal figure, and let the side parallel to AB meet the circumference in E. Join AE, EB, through A draw AF parallel to BE and join BF.

21. The greatest quadrilateral in a circle is a square. By reference to Euc. III. 22, and Theo. 21, p. 252, it will appear when a circle can be inscribed in, and circumscribed about a quadrilateral figure.

22. Bisect the angle contained by the two lines at the point where the bisecting line meets the circumference, draw a tangent to the circle and produce the two straight lines to meet it. In this triangle inscribe a circle.

23. Join the given point and the centre of the circle, and at the point where the circumference is cut, draw a tangent meeting the two other tangents; the circle inscribed in the triangle thus formed, will be the circle required.

24. If ABCD be the required square. Join O, O′ the centres of the circles and draw the diagonal AEC cutting OO′ in E. Then E is the middle point of OO′ and the angle AEO is half a right angle.

25. The line AB joining the points of contact is bisected in D by the line CC′ joining the centres of the circles. If DE, DF be taken each equal to half the given side, the contruction is obvious.

26. Let A, B, C be the three points, join AB, AC, BC.

The point required, will be found to be that point in which three circles circumscribing the equilateral triangles on AB, AC, BC, meet within the triangle.

27. First shew the possibility of a circle circumscribing such a figure, and then determine the centre of the circle.

28. The centre of the circumscribing circle is determined by the intersection of the two lines which bisect the angles adjacent to any side of the quadrilateral figure.

29. Let ABC be a triangle, having C a right angle, and upon AC, BC let semicircles be described: bisect the hypothenuse in D, and let fall DE, DF perpendiculars on AC, BC respectively, and produce them to meet the circumferences of the semicircles in P, Q; then DP may be proved to be equal to DQ.

30. Let O, O′ be the centres of the semicircles on AC and CB.

At C draw CD perpendicular to AB meeting the circumference in D. Produce CD and make DE equal to the radius of either of the smaller semicircles. Join EO, EO′ and let F be the centre of the circle described about the triangle EOO′, join FO, FO′ meeting the circumferences in G, H. Then FD, FG, FH may be proved to be equal to one another.

By Euc. III. 36, twice OG may be shewn to be equal to three times GF.

31. Suppose the centre of the required circle to be found, let fall two perpendiculars from this point upon the radii of the quadrant, and join the centre of the circle with the centre of the quadrant and produce the line to meet the arc of the quadrant. If three tangents be drawn at the three points thus determined in the two semicircles and the arc of the quadrant, they form a right-angled triangle which circumscribes the required circle.

32. Suppose the parallelogram to be rectangular and inscribed in the given triangle and to be equal in area to half the triangle: it may be shewn that the parallelogram is equal to half the altitude of the triangle, and that there is a restriction to the magnitude of the angle which two adjacent sides of the parallelogram make with one another.

33. Let ABC be the given triangle, and A′B′C′ the other triangle, to the sides of which the inscribed triangle is required to be parallel. Through any point *a* in AB draw *ab* parallel to A′B′ one side of the given triangle and through *a*, *b* draw *ac*, *bc* respectively parallel to AC, BC. Join A*c* and produce it to meet BC in D; through D draw DE, DF, parallel to *ca*, *cb*, respectively, and join EF. Then DEF is the triangle required.

34. (1) Let ABCD be the given square: join AC, at A in AC, make the angles CAE, CAF, each equal to one third of a right angle, and join EF.

(2) Bisect AB any side in P, and draw PQ parallel to AD or BC, then at P make the angles as in the former case.

35. Let ABCD be the given square. With centre A and radius AB describe a circle, which will pass through D: take DE equal to DB. With centres B, E and radius BE describe two arcs cutting each other in F. With centre D and radius DF describe a circle cutting the sides AD, DC of the square in G, H. Then B, G, H, are the angular points of the inscribed equilateral triangle.

36. Join AB, and from A draw AT to touch the circle in T. Divide AB in C so that the rectangle contained by AB, AC shall be equal to the square of AT. Through C draw CP to touch the circle in P, join AP and produce it to meet the circumference in X. Draw XB intersecting the circumference in Q and join PQ. PQ is parallel to AB. The proof requires Euc. VI. 6. How is the construction to be effected if the two points A, B be given within the circle?

37. Let AB be the two given points and C the centre of the given circle. Join AB and describe a circle passing through A, B and touching the circle whose centre is C in the point D, join AD, BD: the angle ADB is greater than the angle AEB, subtended by AB at any other point E in the circumference of the circle whose centre is C.

38. The point P will be found to be that point where a circle passing through the extremities of the given line touches the given circle.

39. The point required will be found to be that point at which the line drawn bisecting the radius is perpendicular to it.

40. The point D may be shewn to be that point in the circumference of the circle which passes through the points A, B and touches the line CD in the point D.

41. The point required is the centre of the circle which circumscribes the triangle. See the notes on Euc. III. 20, p. 108.

42. If the perpendiculars meet the three sides of the triangle, the point is within the triangle, Euc. IV. 4. If the perpendiculars meet the base and the two sides produced, the point is the centre of the *escribed* circle.

43. In general three straight lines when produced will meet and form a triangle, except when all three are parallel or two parallel are intersected by the third. This Problem includes Euc. IV. 5 and all the cases which arise from producing the sides of the triangle. The circles described touching a side of a triangle and the other two sides produced, are called the *escribed* circles.

44. Let the two given lines AB, CD when produced meet in E, and let F be the given point. Bisect the angle AEC by EG and through F draw FG perpendicular to EG and produce it both ways to meet AB in B, and CD in D. Take GH equal to GF, and on BA make BK such that the square of BK is equal to the rectangle contained by BH, BF. The circle described through the points K, F, H shall touch the lines AB, CD. If the lines be parallel there is no difficulty in the construction.

45. Let the circle required touch the given circle in P and the given line in Q. Let C be the centre of the given circle and C′ that of the required circle. Join CC′, C′Q, QP; and let QP produced meet the given circle in R, join RC and produce it to meet the given line in V. Then RCV is perpendicular to TQ. Hence the construction.

46. Let A be the centre of the given circle, B the given point in the circumference, and C the other given point. Join AB, BC, and make the angle BCD equal to ABC: then CD meets AB in D, the centre of the required circle.

The given point C may be *inside* or *outside* the given circle, and the contact appears to be always possible.

47. Let A, B be the centres of the given circles and CD the given straight line. On the side of CD opposite to that on which the circles are situated, draw a line EF parallel to CD at a distance equal to the radius of the smaller circle. From A the centre of the larger circle describe a concentric circle GH with radius equal to the difference of the radii of the two circles. Then the centre of the circle touching the circle GH, the line EF, and passing through the centre of the smaller circle B, may be shewn to be the centre of the circle which touches the circles whose centres are A, B, and the line CD.

48. Let AB, CD be the two lines given in position and E the centre of the given circle. Draw two lines FG, HI parallel to AB, CD respectively and external to them. Describe a circle passing through E and touching FG, HI. Join the centres E, O, and with centre O and radius equal to the difference of the radii of these circles describe a circle; this will be the circle required.

49. Let the circle ACF having the centre G, be the required circle touching the given circle whose centre is B, in the point A, and cutting the other given circle in the point C. Join BG, and through A, draw a line perpendicular to BG; then this line is a common tangent to the circles whose centres are B, G. Join AC, GC. Hence the construction.

50. Let A be the given point, BC the given straight line which the circle is to touch, and DE the line in which the centre is to be situated. Let DE be produced to meet BC in C. Join AC and through A draw AB perpendicular to BC, and produce it to meet DEC in D. With centre D and radius DB describe a circle cutting CA, produced if necessary, in two points F, G, or touching it in one. Join FD, and draw AH parallel to FD meeting DE in H. The circle described with centre H and radius HA is the circle required. Draw HK perpendicular to BC, then by similar triangles, HG is proved to be equal to HA.

51. This is a particular case of the general problem; to describe a circle passing through a given point and touching two straight lines given in position.

Let A be the given point between the two given lines which when produced meet in the point B. Bisect the angle at B by BD and through A draw AD perpendicular to BD and produce it to meet the two given lines in C, E. Take DF equal to DA, and on CB take CG such that the rectangle contained by CF, CA is equal to the square of CG. The circle described through the points F, A, G, will be the circle required. Deduce the particular case when the given lines are at right angles to one another, and the given point in the line which bisects the angle at B.

If the lines are parallel, when is the solution possible?

52. Let A, B, be the centres of the given circles, which touch externally in E; and let C be the given point in that whose centre is B. Make CD equal to AE and draw AD; make the angle DAG equal to the angle ADG: then G is the centre of the circle required, and GC its radius.

53. Let C be the given point in the given line AB, and D the centre of the given circle. Through C draw a line CE perpendicular to AB, on the other side of AB, take CE equal to the radius of the given circle. Join ED, and at D make the angle EDF equal to the angle DEC, and produce EC to meet DF in F.

54. If the three points be such as when joined by straight lines a triangle is formed; the points at which the inscribed circle touches the sides of the triangle, are the points at which the three circles touch one another. Euc. IV. 4. There are different cases which arise from the relative position of the three points.

55. Bisect the sides AB, BC, CA, in D, E, F, and join DE, EF, FD; then the circles described about the triangles ADF, BDE, CEF shall pass through the angles A, B, C of the equilateral triangle and shall touch each other.

56. Suppose the triangle constructed, then it may be shewn that the difference between the hypothenuse and the sum of the two sides is equal to the diameter of the inscribed circle.

57. With the given radius of the circumscribed circle, describe a circle. Draw BC cutting off the segment BAC containing an angle equal to the given vertical angle. Bisect BC in D, and draw the diameter EDF: join FB, and with centre F and radius FB describe a circle: this will be the locus of the centres of the inscribed circle (see Theorem 27, p. 339). On DE take DG equal to the given radius of the inscribed circle, and through G draw GH parallel to BC, and meeting the locus of the centres in G. G is the centre of the inscribed circle.

58. This may readily be effected in almost a similar way as the preceding Problem.

59. With the given radius describe a circle, then by Euc. III. 34.

60. Let AD make with a diameter AB (2 R) an angle DAB equal to one-third of a right angle, and let the required circle DHG whose centre is F, and radius $\frac{1}{3}$. R, touch the straight line AD in D and the circle in H. Join FC intersecting the circles at H, and FD, also draw FE parallel to DA, and AE to DF. Then since the circle DHG touches the given line AD (for the angle BAD being given, AD is given), and since its radius FD is given, the locus of the centre is the line EF at the given distance AE from AD. Also since HF and HC are given, CF is given, and C being given, the locus of F is the circle described about C with a radius equal to the sum of the radii HF, HC.

61. Let ABC be a triangle on the given base BC and having its vertical angle A equal to the given angle. Then since the angle at A is constant, A is a point in the arc of a segment of a circle described on BC. Let D be the centre of the circle inscribed in the triangle ABC. Join DA, DB, DC: then the angles at B, C, A, are bisected. Euc. IV. 4. Also since the angles of each of the triangles ABC, DBC are equal to two right angles, it follows that the angle BDC is equal to the angle A and half the sum of the angles B and C. But the sum of the angles B and C can be found because A is given. Hence the angle BDC is known and therefore D is the locus of the vertex of a triangle described on the base BC and having its vertical angle at D double of the angle at A.

62. Let the two circles touch each other in D and the given line in A, B, and let C, C′ be the centres of the circles. Join CA, C′B, C′C, and draw CE parallel to AB meeting AC in E. Hence the construction.

63. Divide the circle into three equal sectors, and draw tangents to the middle points of the arcs, the problem is then reduced to the inscription of a circle in a triangle.

64. The general case of this problem is when the given circles do not *touch* or *intersect* one another. Let A, B, C be the centres of the given circles. With centre B describe a circle with a radius equal to the difference or sum (as the case may require) of the radii of the circles whose centres are A and B: with centre C describe another circle with a radius equal to the difference or sum of the radii of the circles whose centres are A and C. Then the circle described touching these two circles and passing through the point A (Prob. 59, p. 351), will have its centre coincident with the centre of the required circle. Give the analysis of the problem.

65. This is the general case of Problem 63 supra.

66. The problem is the same as to find how many equal circles may be placed round a circle of the same radius, touching this circle and each other. The number is six.

67. This may be effected by Euc. IV. 10.

68. Apply Euc. IV. 10; I. 23; IV. 11.

69. See Problem 27, p. 298.

70. If one of the diagonals be drawn, this line with three sides of the pentagon forms a quadrilateral figure of which three consecutive sides are equal. The problem is reduced to the inscription of a quadrilateral in a square.

71. This may be deduced from Euc. IV. 11.

72. The line AC or BD (fig. Euc. IV. 10) is the side of a regular decagon inscribed in the circle. See the note, Euc. II. 11, p. 72.

73. Each side of the inscribed equilateral triangle subtends an arc equal to one third of the circumference. Hence the method is at once obvious.

74. Let the areas of the inscribed and circumscribed hexagons be expressed in terms of the radius of the circle.

75. The alternate sides of the hexagon will fall upon the sides of the triangle, and each side will be found to be equal to one third of the side of the equilateral triangle.

76. This construction may be effected in two different ways.

(1) When four sides of the hexagon fall upon the sides of the square. If AC be a diagonal of the square ABCD; a line EF may be drawn parallel to AC, by problem 43, p. 298, such that AE, EF, FC shall be equal to one another.

(2) When only two of the sides of the hexagon fall on the sides of the square. Bisect the opposite sides AB, CD in E, F; the problem is then reduced to that of drawing two lines from E, F to meet BC in G, H, such that EG, GH, HF shall be equal to one another.

77. A regular duodecagon may be inscribed in a circle by means of the equilateral triangle and square, or by means of the hexagon. If $r$ be the radius of the circle, the area of the duodecagon is $3r^2$, which is the square of the side of an equilateral triangle inscribed in the same circle. Theorem 1, p. 332.

78. By Euc. I. 47, the perpendicular distance from the centre of the circle upon the side of the inscribed hexagon may be found. The comparison of the areas of the two figures may be made by comparing the sums of the areas of the triangles by which they are respectively formed, by drawing lines from the angular points of the figures to the centre of the circle which circumscribes the figures.

79. If the pentagon be equilateral and equiangular, the problem is impossible; it is however possible to inscribe a regular hexagon in an irregular pentagon, when the pentagon admits of an inscribed circle, and the points of contact are five of the points of the inscribed hexagon.

80. Each of the interior angles of a regular octagon may be shewn to be equal to three-fourths of two right angles, and the exterior angles made by producing the sides, are each equal to one-fourth of two right angles, or one half of a right angle.

81. This is found from the inscribed square.

82. If the alternate sides of the octagon be produced to meet one another, the figure thus formed is a square, and the area of the octagon may be shewn to be the difference between the area of the square, and twice the square of the side of the octagon.

83. Let the areas of the inscribed hexagon and the circumscribed octagon be expressed in terms of the radius of the circle.

84. The pentagon may be transformed into a square, and then the problem is to describe a regular octagon equal in area to a given square.

85. The value of the interior angle of any regular figure may be found by means of the note on Euc. IV. 16, p. 125.

86. If the least angle be denoted by $\theta$, the other angles are $\theta+10$, $\theta+20$, &c. degrees; by applying the expression for the sum of an arithmetical series, and note p. 125, $\theta$ will be found to be $99^\circ$.

87. Let $n$ be the number of sides; then the sum of the interior angles of the figure may be found by finding the sum of an arithmetic progression of $n$ terms, whose first term is $120^\circ$, and common difference $5^\circ$: and by note on Euc. IV. 16, p. 125, the number of sides will be found to be 16 and 9. Construct the two figures, and shew that one of them contains re-entrant angles.

88. Proceed as in Problem 87.

89. Proceed as in Problem 87.

90. Proceed as in Problem 87.

91. The number of sides may be found by the note on Euc. IV. 16, p. 125.

92. By means of the note p. 125, the figure may be shewn to be a regular nonagon.

93. The same method as in Euc. IV. 14, may be employed for determining the centre of the circle which will circumscribe any regular polygon.

94. This may be effected by the same construction as Problem 27, p. 298.

95. Every regular polygon can be divided into equal isosceles triangles by drawing lines from the centre of the inscribed or circumscribed circle to the angular points of the figure, and the number of triangles will be equal to the number of sides of the polygon. If a perpendicular FG be let fall from F (figure Euc. IV. 14,) the centre on the base CD of FCD, one of these triangles, and if GF be produced to H till FH be equal to FG, and HC, HD be joined, an isosceles triangle is formed, such that the angle at H is half the angle at F. Bisect HC, HD in K, L, and join KL; then the triangle HKL may be placed round the vertex H, twice as many times as the triangle CFD round the vertex F.

96. Each of the vertical angles of the triangles so formed, may be proved to be equal to the difference between the exterior and interior angle of the heptagon.

97. See note on Euc. IV. 16, p. 125.

98. See Euc. I. 9, note p. 49; Problems 10, 11, p. 297; Euc. IV. 10; Euc. IV. 16, note p. 124; Problem 67, p. 336.

99. The equilateral triangle can be proved to be the least triangle which can be circumscribed about a circle.

100. Let ABC be the equilateral triangle, and let $a$, $b$, $c$ be the centres of the squares described upon the sides opposite to the angles A, B, C respectively. The triangle formed by drawing $ab$, $bc$, $ca$ may be proved to be an equilateral triangle, the sides of which are respectively equal to the line drawn from any angle of the given triangle to the centre of the square on the opposite side. If the numerical value of the side of the given triangle be given, the areas of the two triangles may be expressed in terms of the given side.

101. The area of the triangle formed by joining the centres, may be shewn to be four times the area of the triangle formed by joining the points of contact of the circles.

102. If the radius of the given circle be unity, the radius of each of the four equal circles which touch it externally and each other, may be shewn to be numerically equal to $1+\sqrt{2}$.

103. Take half of the side of the square inscribed in the given circle, this will be equal to a side of the required octagon. At the extremities on the same side of this line make two angles each equal to three-fourths of two right angles, bisect these angles by two straight lines, the point at which they meet will be the centre of the circle which circumscribes the octagon, and either of the bisecting lines is the radius of the circle.

104. (1) When $n=2$, the figure is a square. (2) When $n=4$, the figure is *any* triangle; and we have only to bisect the sides, which will be the points at which the inscribed figure has its angles situated. In all cases except the triangle, the given figure must be equilateral and equiangular.

## HINTS, &c. TO THE THEOREMS. BOOK IV.

2. SEE Euc. IV. 4, 5. The centres of the two circles may be proved to coincide, and the diameter of the circumscribed circle may be shewn to be double of the diameter of the inscribed circle.

3. Let the figure be constructed, the sum of the sides of one triangle may be proved to be double the sum of the sides of the other: and the area of one, four times the area of the other. The parallelism of the lines is proved by Euc. III. 32; I. 29.

4. The line joining the points of bisection, is parallel to the base of the triangle and therefore cuts off an equilateral triangle from the given triangle. By Euc. III. 21; I. 6, the truth of the theorem may be shewn.

5. See Theorem 22, p. 301.

6. Prove $aBc$, $cAb$, $bCa$ to be straight lines, and the angles at $a$, $b$, $c$ equal.

7. Let three equilateral triangles be described upon AB, AC, BC, the sides of any triangle, and let D, E, F be the centres of the circles inscribed in the equilateral triangles on AB, AC, BC respectively. Let DE, EF, FD be drawn; then EFD is an equilateral triangle. Join DA, DB, EA, EC, FB, FC. At E in AE make the angle AEG equal to FEC, and take EG equal to ED, and join GA. Then the angles of the triangles GDE, DEF may be proved to be respectively equal, and each equal to two-thirds of two right angles.

8. Let the line AD drawn from the vertex A of the equilateral triangle, cut the base BC, and meet the circumference of the circle in D. Let DB, DC be joined: AD is equal to DB and DC. If on DA, DE be taken equal to DB, and BE be joined; BDE may be proved to be an equilateral triangle, also the triangle ABE may be proved equal to the triangle CBD.

The other case is when the line does not cut the base.

9. Let the figure be described. Join DC, then DC is a diameter of the circle described about the quadrilateral figure CFDE. Bisect DC in G, and join FG. If FS can be proved to be perpendicular to FG, then FS will be a tangent to the circle at F, Euc. III. 18.

10. Let ABC be an equilateral triangle inscribed in a circle, and let AB′C′ be an isosceles triangle inscribed in the same circle, having the same vertex A. Draw the diameter AD intersecting BC in E, and B′C′ in E′, and let B′C′ fall below BC. Then AB, BE, and AB′, B′E′, are respectively the semi-perimeters of the triangles. Draw B′F perpendicular to BC, and cut off AH equal to AB, and join BH. If BF can be proved to be greater than B′H, the perimeter of ABC is greater than the perimeter of AB′C′. Next let B′C′ fall above BC.

11. Let the equilateral triangle ABC whose altitude is AD, be turned round its centre O till it assume the position *abc*, and let the base *bc* of the new position cut BC in E. Produce *ad* to meet BC in F. Then from the right-angled triangles ODF, *d*EF, the angle between the two positions of the altitude is proved to be equal to the angle between the bases BC, *bc*.

12. Let a diameter be drawn from any angle of an equilateral triangle inscribed in a circle, to meet the circumference. It may be proved that the radius is bisected by the opposite side of the triangle.

13. Let a circle be described upon the base of the equilateral triangle, and let an equilateral triangle be inscribed in the circle. Draw a diameter from one of the vertices of the inscribed triangle, and join the other extremity of the diameter with one of the other extremities of the sides of the inscribed triangle. The side of the inscribed triangle may then be proved equal to the perpendicular in the other triangle.

14. Let the angle BAC be a right angle, fig. Euc. IV. 4. Join AD. Then Euc. III. 17, note p. 108.

15. By the preceding theorem, the excess of the two sides containing the right angle above the hypothenuse is equal to the diameter of the inscribed circle. In this theorem the hypothenuse is equal to the diameter of the circumscribed circle.

16. Let P, Q be the middle points of the arcs AB, AC, and let PQ be joined, cutting AB, AC in D, E; then AD is equal to AE. Find the centre O, and join OP, QO.

17. Let the figure be constructed; the proof depends on Theorem 3, p. 313.

18. Let ABC be any triangle inscribed in a circle, and let the perpendiculars AD, BE, CF intersect in G. Produce AD to meet the circumference in H, and join BH, CH. Then the triangle BHC may be shewn to be equal in all respects to the triangle BGC, and the circle which circumscribes one of the triangles will also circumscribe the other. Similarly may be shewn, by producing BE and CF, &c.

19. Let ABC be a triangle, F the centre of the circumscribed circle (figure Euc. IV. 5,) FD, FE, FG, the perpendiculars from F on AB, AC, BC respectively. Draw DE, DG, GE. Then each of the quadrilaterals ADFE, BDFG, GFEC may be circumscribed by a circle, Euc. III. 22. Then by Euc. VI. E, and observing that twice the area of the triangle ABC is equal to the sum of the rectangles contained by the perpendiculars FD, FE, FG and the sides on which they respectively fall, and also to the rectangle by the sum of the sides and the radius of the inscribed circle, we may shew that the rectangle contained by the sum of the perpendiculars and the sum of the sides of the triangle, is equal to the rectangle contained by the sum of the sides, and the sum of the radii of the inscribed and circumscribed circles.

20. Let F, G, figure, Euc. IV. 5, be the centres of the circumscribed and inscribed circles; join GF, GA, then the angle GAF which is equal to the difference of the angles GAD, FAD, may be shewn equal to half the difference of the angles ABC and ACB.

21. Through C draw CH parallel to AB and join AH. Then HAC the differ-

ence of the angles at the base is equal to the angle HFC, Euc. III. 21, and HFC is bisected by FG.

22. In the figure, Euc. IV. 5. Let AF bisect the angle at A, and be produced to meet the circumference in G. Join GB, GC and find the centre H of the circle inscribed in the triangle ABC. The lines GH, GB, GC are equal to one another.

23. This property of the figure, Euc. IV. 4, exhibited by joining AD and producing it to meet the base, is shewn to follow from Euc. I. 32.

24. This is manifest from Euc. III. 21.

25. This is manifest from Euc. III. 11, 18.

26. The proof of this Theorem is contained in that of Problem 30, p. 334.

27. This Theorem may be stated more generally as follows:

Let AB be the base of a triangle, AEB the locus of the vertex; D the bisection of the remaining arc ADB of the circumscribing circle: then the locus of the centre of the inscribed circle is another circle whose centre is D and radius DB. For join CD: then P the centre of the inscribed circle is in CD. Join AP, PB; then these lines bisect the angles CAB, CBA, and DB, DP, DA may be proved to be equal to one another.

28. Let a circle inscribed in the triangle AED, (figure, Theorem 3, p. 113,) touch the base ED in H, and the sides AD, AE in K, L respectively. It may be easily shewn that EF is equal to DH, and BL or CK equal to ED.

29. The base BC is intersected by the perpendicular AD, and the side AC is intersected by the perpendicular BE. From Theorem 4, p. 314, the arc AF is proved equal to AE, or the arc FE is bisected in A. In the same manner may the arcs FD, DE be shewn to be bisected in B, C.

30. Let ABC be a triangle, and let D, E be the points where the inscribed circle touches the sides AB, AC. Draw BE, CD intersecting each other in O. Join AO, and produce it to meet BC in F. Then F is the point where the inscribed circle touches the third side BC. If F be not the point of contact, let some other point G be the point of contact. Through D draw DH parallel to AC, and DK parallel to BC. By the similar triangles, CG may be proved equal to CF, or G the point of contact coincides with F, the point where the line drawn from A through O meets BC.

31. The difference of the two squares is obviously the sum of the four triangles at the corners of the exterior square.

32. The areas of the successive inscribed squares will be found to be respectively, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c., of the given square, and the sum of these squares may be found by finding the sum of the series $\frac{1}{2}$, $\frac{1}{4}$, &c.

33. Let the squares be inscribed in, and circumscribed about a circle, and let the diameters be drawn, the relation of the two squares is manifest.

34. The opposite angles of every quadrilateral about which a circle can be described are together equal to two right angles: and as the opposite angles of every parallelogram are equal to one another, it is obvious that no circle can be described about a parallelogram except it be rectangular.

35. Let one of the diagonals of the square be drawn, then the isosceles right-angled triangle which is half the square, may be proved greater than any other right-angled triangle upon the same hypothenuse.

36. This may be made to appear from Euc. III. 21, 27.

37. The four lines drawn from the centre of the circle to the angular points of the quadrilateral divide the circumference of the circle into four parts, the sum of each pair of opposite portions may be proved equal to half the circumference.

38. This will be manifest from the equality of the two tangents drawn to a circle from the same point.

39. This is the converse proposition of the preceding, and its truth may be proved indirectly, after having proved the direct form of the proposition.

40. Let AC, BD be joined in the figure, Euc. III. 14; the sides AC, BD are parallel. Join AD, BC. Then since the angles in equal segments of the same circle are equal, Euc. I. 27.

In the same way the opposite sides may be shewn to be parallel when the quadrilateral figure circumscribes the circle.

41. Let ABCD, A′B′C′D′ be two quadrilateral figures having their corresponding sides respectively equal, and let ABCD be circumscribed by a circle, and A′B′C′D′

not capable of being so circumscribed, the quadrilateral ABCD is greater than the quadrilateral A'B'C'D'. Draw the diameter AE, intersecting the side CD, and join CE, ED. On C'D' describe the triangle C'E'D' having its sides respectively equal to the sides of the triangle CED, and join A'E'. The proof depends on shewing that of the quadrilaterals ABCE, A'B'C'E', which have their sides respectively equal, the greater is that which has its fourth side equal to the diameter of the circumscribing circle.

42. This follows as a corollary from Theorem 45, infra.

43. Let the sides of the quadrilateral be produced, and the four circles described touching one side of the figure and the two adjacent sides produced. If two lines be drawn from each exterior angle to the centres of the circles adjacent, the figure so formed may be proved to be a quadrilateral having the sum of each pair of its opposite angles equal to two right angles.

44. Let the diagonal AD cut the arc in P, and let O be the centre of the inscribed circle. Draw OQ perpendicular to AB. Draw PE a tangent at P meeting AB produced in E: then BE is equal to PD. Join PQ, PB. Then AB may be proved equal to QE. Hence AQ is equal to BE or DP.

45. Let A, B, C, D be the angular points of the inscribed quadrilateral, and E, F, G, H those of the circumscribed one whose points of contact with the circle are at A, B, C, D: it is required to prove that the diagonals AC, BD, EF, GH intersect in one point.

If EF do not cut the diagonals AC, BD in the same point, let it cut AC in Q, and BD in Q'; and draw EK parallel to FC, and EL to FD. Then AE, EK, EB, EL may be shewn to be equal to one another. Again, since EF cuts AC in Q, and BD in Q', by Euc. VI. 4. KE : FC :: QE : QF; and LE : DF :: Q'E : Q'F; and since KE = LE, and FC = DF, therefore QE : QF :: Q'E : Q'F, which is impossible. Whence FE does not cut AC, BD in different points: that is, it passes through their intersection. In the same manner GH is shewn to pass through the intersection of AC, BD.

(2) If the point of intersection be *without* the circle, the same reasoning applies, with the single exception, that EAC and AKE are respectively the supplements of ACG instead of equal to it.

(3) It therefore follows that AB and CD will intersect in a point in GH, and that BC, AD will intersect in EF.

46. Let a circle inscribed in a square touch one of the sides in the point A, let ABC be an equilateral triangle inscribed in the circle, also let a circle inscribed in the triangle, touch BC, CA, AB in the points D, E, F respectively. AD being joined passes through O the common centre of the two circles. If any point P be taken in the circumference of the inner circle, and PA, PB, PC be drawn, then $PA^2 + PB^2 + PC^2 + OD^2 = 4 . AO^2$, or the area of the given square. By theorem 22, page 309, after joining PF, PC, PD: $PA^2 + PB^2 + PC^2 = PD^2 + PE^2 + PF^2 + 3 . BF^2$. If perpendiculars PL, PM, PN be drawn to the sides of the triangle BC, CA, AB; since the square of the chord PD drawn from any point P to the point of contact of the inscribed circle, is equal to the rectangle contained by the perpendicular PL and the diameter of the circle, $PD^2 + PE^2 + PF^2 = (PL + PM + PN) . 2DO$; also $PL + PM + PN = AD$ (Theorem 15, p. 308) $= 3 . OD$ (Euc. IV. 4). Whence may be deduced the required property.

Is the property true when the point P is not on the circumference, but any point within the circle?

47. This point will be found to be the intersection of the diagonals of the given parallelogram.

48. This theorem is the converse of part of Theorem 31, p. 340, but including also the rhombus as well as the square.

49. Let the inscribed circles whose centres are A, B touch each other in G, and the circle whose centre is C, in the points D, E; join A, D; A, E; at D, draw DF perpendicular to DA, and EF to EB, meeting in F. Let F, G be joined, and FG be proved to touch the two circles in G whose centres are A and B.

50. This is obvious from Euc. IV. 7, the side of a square circumscribing a circle being equal to the diameter of the circle.

51. Let a diameter be drawn from P through the centre Q, and join PA', PB',

PC′, also from A′, B′, C′ draw lines perpendicular to the diameter through P. Then PA′Q, PB′Q, PC′Q are three triangles, of which one will be obtuse-angled and the other two acute-angled. Then by Euc. II. 12, 13, $PA'^2+PB'^2+PC'^2$ may be found to be equal to $3.PQ^2+3.A'Q^2$. Again, by joining P′A, P′B, P′C, &c. in a similar way, $P'A^2+P'B^2+P'C^2$ may be found to be equal to the same quantity.

52. If BD be shewn to subtend an arc of the larger circle equal to one tenth of the whole circumference:—then BD is a side of the decagon in the larger circle. And if the triangle ABD can be shewn to be inscriptible in the smaller circle, BD will be the side of the inscribed pentagon.

53. It may be shewn that the angles ABF, BFD stand on two arcs, one of which is three times as large as the other.

54. It may be proved that the diagonals bisect the angles of the pentagon; and the fivesided figure formed by their intersection may be shewn to be both equiangular and equilateral.

55. The figure ABCDE is an irregular pentagon inscribed in a circle; it may be shewn that the five angles at the circumference stand upon arcs whose sum is equal to the whole circumference of the circle; Euc. III. 20.

56. Prove the five lines joining the points of intersection to be equal to one another, and the angles contained by every two lines which are adjacent to one another.

57. This is a modified form of stating one of the properties in Theorem 64, p. 341.

58. The figure may be proved to be both equilateral and equiangular by means of the isosceles triangles formed by producing the sides of the pentagon.

59. The angles at $\alpha$, $\beta$, $\gamma$, $\delta$, $\epsilon$ may be proved equal to one another, and each equal to four-fifths of a right angle.

60. If a side CD (figure, Euc. IV. 11) of a regular pentagon be produced to K, the exterior angle ADK of the inscribed quadrilateral figure ABCD is equal to the angle ABC one of the interior angles of the pentagon. From this a construction may be made for the method of folding the ribbon.

61. Let AP be drawn from A perpendicular on CD, figure, Euc. IV. 11. AP may be proved to pass through the centre of the circumscribing circle, as also the perpendiculars from the other angles of the figure. The relation of the angles may be found by means of Euc. I. 32. Cor.

62. The sides and diagonals of the two pentagons (see figure, Euc. IV. 11) may be shewn to have the same relation to each other which is proved in Euc. IV. 10.

63. By Euc. II. 11, $FO^2=AF.FO$, FO is the side of the regular decagon inscribed in the circle, and OC is the side of the hexagon. Also, $CF^2=CO^2+OF^2$, and by Theorem 39, p. 355, CF is a side of the pentagon inscribed in the circle.

64. In the figure, Euc. IV. 10. Let DC be produced to meet the circumference in F, and join FB. Then FB is the side of a regular pentagon inscribed in the larger circle, D is the middle of the arc subtended by the adjacent side of the pentagon. Then the difference of FD and BD is equal to the radius AB. Next, it may be shewn, that FD is divided in the same manner in C as AB, and by Euc. II. 4, 11, the squares of FD and DB are three times the square of AB, and the rectangle of FD and DB is equal to the square of AB.

65. Each of the figures thus formed exterior to the hexagon is an equilateral triangle.

66. The angles contained in the two segments of the circle, may be shewn to be equal, then by joining the extremities of the arcs, the two remaining sides may be shewn to be parallel.

67. It may be shewn that four equal and equilateral triangles will form an equilateral triangle of the same perimeter as the hexagon, which is formed by six equal and equilateral triangles.

68. Let the alternate sides of the figure, Euc. IV. 15, be produced to meet; each of the triangles so formed exterior to the hexagon, may be proved equal in all respects to each of the six triangles into which the hexagon is divided by the diagonals.

69. Let the figure be constructed. By drawing the diagonals of the hexagon, the proof is obvious.

70. Let the figure be drawn, O being the centre and A, B any two opposite angles of the hexagon. If AP, BP′, OQ be the perpendiculars on the line, then the

sum of AP and BP′ may be proved equal to twice OQ, and in a similar way the sum of the pairs of perpendiculars from the two remaining pairs of opposite angles.

71. The circles will be the escribed circles of the six triangles formed by joining the centre of the circle and the angular points of the circumscribed hexagon.

72. This appears directly from Euc. I. 38; and IV. 15.

73. The square of the tangent, by Euc. III. 36, and the square of the side of the octagon may be shewn each to be equal to the same quantity, $(2-\sqrt{2})r^2$, where $r$ is the radius of the circle.

74. By constructing the figures and drawing lines from the centre of the circle to the angles of the octagon, the areas of the eight triangles may be easily shewn to be equal to eight times the rectangle contained by the radius of the circle, and half the side of the inscribed square.

75. Let a regular polygon ABCDE be taken, O being the centre of the circumscribed circle, and E the bisection of the side DC opposite to the angle A. Join AO, OE, and prove that AO, OE are in the same straight line.

76. The first part may be proved by Euc. III. 21. The converse, when the number of sides is *odd*, follows, and may be proved ex absurdo. But when the number of sides is *even*, every pair of opposite angles may be equal, as in the case of the rectangle, which has all its angles equal, but not all its sides equal.

77. Let ABCDEF be a figure of six sides, having all its interior angles equal to one another, but not its opposite sides equal. Produce AB, DC to meet in G, and BC, ED to meet in H. Then AB may be shewn to be parallel to ED.

78. Let lines be drawn to the centre of the circle from the extremities of the lines and the points of contact, and the loci of the extremities of the lines may be shewn to be in the circumferences of two concentric circles, unless the parts of the lines on each side of the points of contact be equal.

79. Join the points of bisection of the equal lines and the centre of the given circle.

80. In the pentagon, hexagon, &c., by Euc. I. 32, the truth of the property is proved.

81. This may be readily shewn in the case of two polygons, one regular, and the other irregular, both having the same number of sides.

82. Let ABCDE (figure Euc. IV. 11,) be a regular polygon inscribed in a circle, and in the arc AB take any point M and join AM, CM. Then the triangle ABC may be shewn to be greater than the triangle AMC.

83. This Theorem is the general form of Theorem 41, p. 340, and may be proved in a similar manner for a polygon of five, six, &c., sides.

84. The proof of this property depends on the fact, that an isosceles triangle has a greater area than any scalene triangle of the same perimeter.

85. The sum of the arcs on which stand the 1st, 3rd, 5th, &c. angles, is equal to the sum of the arcs on which stand the 2nd, 4th, 6th, &c. angles.

86. Let two lines also be drawn from the centre of the circumscribed circle to the extremities of the same side.

87. Let ABCDEF be any irregular polygon of six sides inscribed in a circle, and let AC, CE, EA, BD, DF, FB be joined intersecting each other in $a, b, c, d, e, f$; then $abcdef$ is an irregular polygon of six sides; and if AC be intersected in $a, b$, then the sum of the interior angles at $a, b$, of the inner polygon, may be shewn to be equal to the sum of the interior angles at A, C of the exterior polygon. Euc. I. 32; III. 21. Suppose the sides of the exterior polygon to be equal, the sides of the interior polygon may readily be shewn to be also equal.

88. The opposite angles of the figure so constructed may be proved to be equal to two right angles. Euc. III. 22.

89. No Geometrical method is known whereby the circumference of a circle can be divided into *seven* or *eleven* equal parts.

90. From the given point draw lines to the angles of the polygon; then the area of the polygon is equal to the areas of the triangles thus formed, namely, the rectangle contained by the sum of the perpendiculars and half one of the equal sides. But the area of the polygon is also equal to $n$ times the area of the equal triangles of which the polygon is composed, that is, $n$ times the rectangle contained by the radius of the inscribed circle and half one side. Hence, &c.

91. This Theorem is the same as the preceding, and forms *Prop.* III. *of Stewart's General Theorems.*

92. Let ABCDEF be a regular hexagon, P, Q, R, S, T, V the points of contact of the inscribed circle, and from any point G let GH, GK, GL, GM, GN, GO be drawn perpendicular to the sides of the figure.

Draw PS, QT, RV intersecting each other in $a$ the centre of the circle, and join GP, GQ, GR, GS, GT, GV. Draw GX, GY, GZ perpendicular to PS, QT, RV and join $Ga$. Then all the angles at $a$ are equal. By Euc. I. 47, $GH^2 + GX^2 = GP^2$, &c.

$$\text{Hence } GH^2 + GK^2 + GL^2 + GM^2 + GN^2 + GO^2 + 2(GX^2 + GY^2 + GZ^2)$$
$$= GP^2 + GQ^2 + GR^2 + GS^2 + GT^2 + GV^2.$$

$$\text{Next shew that } GP^2 + GQ^2 + \&c. = 6 \,.\, Ga^2 + 6 \,.\, Pa^2 = 6 \,.\, d^2 + 6 \,.\, r^2.$$

The points X, Y, Z, are in the circumference of a circle whose diameter is $Ga$, and the circumference is divided into equal parts in X, Y, Z, also $b$, the bisection of $Ga$, is the centre of this circle, and $GX^2 + GY^2 + GZ^2 = 3 \,.\, Gb^2$. Whence may be shewn, that $GH^2 + GK^2 + \&c. = 6\left(\frac{d^2}{2} + r^2\right)$.

Prove the property, when the figure is a regular pentagon.

This is *Prop.* V. *of Stewart's General Theorems.*

93. This proposition is more easily established by the method of co-ordinates than by pure Geometry. In this way it has been proved by Dr Wallace, in the Gentleman's Mathematical Companion, Vol. VI. p. 452.

The case of the triangle has been proved Geometrically by Mr Kay, in Leybourn's Mathematical Repository, (NS.) Vol. III. p. 35, and Trigonometrically also by Dr Wallace in the same place. A complete Geometrical demonstration, however, may be obtained by means of a general method of investigating certain classes of properties of the circle given by Lieut. Glenie, of the Royal Engineers, in Vol. VI. of the Edinburgh Transactions.

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## HINTS, &c. TO THE PROBLEMS. BOOK VI.

4. Let AB be the given perimeter of the required triangle. On AB describe a triangle ABC similar to the given triangle; bisect the angles at A and B by lines meeting in D; through D, draw DE, DF parallel to AC, BC, and meeting AB in E, F: then DEF is the triangle required.

5. On any side BC of the given triangle ABC, take BD equal to the given base; join AD, through C draw CE parallel to AD, meeting BA produced, if necessary, in E, join ED: then BDE is the triangle required.

6. (1) In every right-angled triangle when its three sides are in Arithmetical progression, they may be shewn to be as the numbers 5, 4, 3. On the given line AC describe a triangle having its sides AC, AD, DC in this proportion, bisect the angles at A, C by AE, CE meeting in E, and through E draw EF, EG parallel to AD, DC meeting in F and G.

(2) Let AC be the sum of the sides of the triangle, fig. Euc. VI. 13. Upon AC describe a triangle ADC whose sides shall be in continued proportion, (by Prob. 9, infra.) Bisect the angles at A and C by two lines meeting in E. From E draw EF, EG parallel to DA, DC respectively.

7. Describe a circle with any radius, and draw within it the straight line MN cutting off a segment containing an angle equal to the given angle, Euc. III. 34. Divide MN in P in the given ratio, and at P draw PA perpendicular to MN and meeting the circumference in A. Join AM, AN, and on AP or AP produced, take AD equal to the given perpendicular, and through D draw BC parallel to MN meeting AM, AN or these lines produced. Then ABC shall be the triangle required.

8. Let A, B be the two given points, and let P be a point in the locus so that PA, PB being joined, PA is to PB in the given ratio. Join AB and divide it in C in the given ratio, and join PC. Then PC bisects the angle APB. Euc. VI. 3.

Again, in AB produced, take AD to AB in the given ratio, join PD and produce AP to E, then PD bisects the angle BPE. Euc. VI. A. Whence CPD is a right angle, and the point P lies in the circumference of a circle whose diameter is CD.

9. By Euc. I. 47. $BC^2 - BE^2 = AC^2 - AE^2$, hence $BC^2 - AC^2 = BE^2 - AE^2$, or $(BC + AC).(BC - AC) = (BE + AE).(BE - AE)$, but $BE - AE = 2.DE$, also $BC - AC = DE$. $\therefore BC + AC = 2.AB$, or the three sides AC, AB, BC are in Arithmetical progression.

10. Let PAQ be the given angle, bisect the angle A by AB, in AB find D the centre of the inscribed circle, and draw DC perpendicular to AP. In DB take DE such that the rectangle DE, DC is equal to the given rectangle. Describe a circle on DE as diameter meeting AP in F, G; and AQ in F′, G′. Join FG′ and AFG′ will be the triangle. Draw DH perpendicular to FG′, and join G′D. By Euc. VI. C, the rectangle FD, DG′ is equal to the rectangle ED, DK or CD, DE.

11. Let BC be the given base; draw BE perpendicular to BC and equal to the given altitude, and through E draw EM parallel to BC. At B make the angle CBF equal to the difference of the angles at the base. Divide the base in D so that BD may be to DC in the ratio of the sides, draw DG perpendicular to BF and produce it to meet EM in A. Join AB, AC; ABC is the triangle required.

12. Let AB be the given base, ACB the segment containing the vertical angle; draw the diameter AD of the circle, and divide it in E in the given ratio; on AE as a diameter, describe a circle AFE; and with centre B and a radius equal to the given line, describe a circle cutting AFE in F. Then AF being drawn and produced to meet the circumscribing circle in C, and CD being joined, ABC is the triangle required. For AF is to FC in the given ratio.

13. Let AC be the given base, and let DAC be the required triangle. Draw DB perpendicular to BC. Then from the hypothesis combined with Euc. VI. 8, it may be shewn that AB is equal to DC, and that AC is divided in B in extreme and mean ratio.

14. Let ABC be any triangle, and DEF the given triangle to which the inscribed triangle is required to be similar. Draw any line $de$ terminated by AB, AC, and on $de$ towards AC describe the triangle $def$ similar to DEF, join $Bf$, and produce it to meet AC in F′. Through F′ draw F′D′ parallel to $fd$, F′E′ parallel to $fe$, and join D′E′, then the triangle D′E′F′ is similar to DEF.

15. Employ Theorem 17, p. 353, and the construction becomes obvious.

16. Let ABC be the required triangle, CD the line bisecting the vertical angle, cutting AB in H, and meeting the circumscribed circle in D, DME a diameter drawn through D, and therefore bisecting the base AB in M. Let DE be bisected in O, then O is the centre of the circumscribing circle; also let P be the centre of the inscribed circle.

Then the line OP joining the centres, cuts the line bisecting the vertical angle in the centre of the inscribed circle. We have given, therefore, the base AB, the segment containing the vertical angle ACB, and the ratio CP to PH.

By means of the equiangular triangles DBC, DHB, the ratio of DB to DC may be shewn to be the same as the ratio of PC to PH, which is given. Hence the ratio of DB to DC, is given; also DB is given in magnitude, and therefore DC. Whence the construction.

17. This is similar in its construction to Prob. 12, supra, except that the point F, instead of being at a given distance from B, is in a semicircle on AB.

18. The line CD is not necessarily parallel to AB. Divide the base AB in C, so that AC is to CB in the ratio of the sides of the triangle.

Then if a point E in CD can be determined such that when AE, CE, EB, are joined, the angle AEB is bisected by CE, the problem is solved.

19. Let ABC be any triangle having the base BC. On the same base describe an isosceles triangle DBC equal to the given triangle. Bisect BC in E, and join DE, also upon BC describe an equilateral triangle. On FD, FB, take EG to EH as EF to EB: also take EK equal to EH and join GH, GK; then GHK is an equilateral triangle equal to the triangle ABC.

20. Let ABC be the required triangle, BC the hypothenuse, and FHKG the inscribed square; the side HK being on BC. Then BC may be proved to be divided in H and K, so that HK is a mean proportional between BH and KC.

21. Let AB be the given perimeter, and let CD be drawn parallel to AB at the distance of the given perpendicular; on AB describe a circle, and let F be the middle point of the semicircle on the opposite side of AB from CD; with centre F describe a circle through A, B, cutting CD in P or P′; make in this circle the arcs AG, BH, equal to BP, PA respectively; and draw PG, PH cutting AB in Q, R: then PQR is the triangle required.

Algebraically. Let $x$, $y$, be the sides, $z$ the hypothenuse of the triangle, and $p$ the perpendicular from the right angle on the hypothenuse: then $x, y, z$, may be found from the equations $x^2+y^2=z^2$, $xy=pz$, $x+y+z=a$, in terms of $p$ the perpendicular, and $a$ the perimeter.

22. Make an isosceles triangle, having its vertical angle equal to the given angle. Describe a triangle similar to this isosceles triangle, and having its perimeter equal to the given perimeter. Then the area of this triangle may be shewn to be greater than the area of any other triangle which has the same vertical angle and the same perimeter.

23. Let ABC be the given triangle. On BC take BD equal to one of the given lines, through A, draw AE parallel to BC. From B draw BE to meet AE in E, and such that BE is a fourth proportional to BC, BD, and the other given line. Join EC, produce BE to F, making BF equal to the other given line, and join FD: then FBD is the triangle required.

24. If a circle be described about the given triangle, and another circle upon the radius drawn from the vertex of the triangle to the centre of the circle, as a diameter, this circle will cut the base in two points, and give two solutions of the problem. Give the Analysis.

25. In the figure, Euc. VI. 13. If E be the middle point of AC; then AE or EC is the arithmetic mean, and DB is the geometric mean, between AB and BC. If DE be joined and BF be drawn perpendicular on DE; then DF may be proved to be the harmonic mean between AB and BC.

26. The two means and the two extremes form an arithmetic series of four lines whose successive differences are equal: the difference therefore between the first and the fourth, or the extremes, is treble the difference between the first and the second.

27. Let the two given lines meet when produced in A. At A draw AD perpendicular to AB, and AE to AC, and such that AD is to AE in the given ratio. Through D, E, draw DF, EF, respectively parallel to AB, AC and meeting each other in F. Join AF and produce it, and the perpendiculars drawn from any point of this line on the two given lines will always be in the given ratio.

28. This problem may be constructed in the same way under more general circumstances than those in which it is enunciated; namely, when A, B, G, are any points, and any line is substituted for BK, compatible with the construction following.

Bisect AB in V and join VG, produce VG to P, make GP = 2 . VG; on PG describe a circle, in which place the chords PQ, PQ′ equal to the given sum of the perpendiculars: then the line QG is that required, and Q′G is that upon which the difference (instead of the sum) of the perpendiculars shall be equal to PG. The proof depends on Prob. 3, p. 347.

29. Let the three given lines meet in A, B, C and form a triangle, and let the ratios of the three perpendiculars be as three lines $m$, $n$, $p$. On AC, BC, take AD, BE each equal to $m$, draw DF, EG each equal to $m$ and parallel to AB. Join AF, BG and produce them to meet in O, the perpendiculars OP, OQ, OR from O drawn to AB, AC, BC, shall have the same ratios as $m$, $n$, $p$. From F draw FH perpendicular to AB, FK to AC, and FL parallel to AC. Then by the similar triangles.

30. Let AB be the given straight line, and let a perpendicular be drawn to AB from the point C. Divide AB in D, so that AD is to DB in the given ratio; then if from D a line DE be drawn to meet the perpendicular in E so that when AE, EB are joined, the angle AEB shall be bisected by ED, E will be the point required. Euc. VI. 3.

31. In BC produced take CE a third proportional to BC and AC; on CE describe a circle, the centre being O; draw the tangent EF at E equal to AC; draw FO cutting the circle in T and T′; and lastly, draw tangents at T, T′ meeting BC in P and P′. These points fulfil the conditions of the problem.

By combining the proportion in the construction with that from the similar triangles ABC, DBP, and Euc. III. 36, 37; it may be proved that CA . PD = $CP^2$.

The demonstration is similar for P'D'.

32. Let ABC be any triangle, and D the given point in the base BC. Divide DC in E so that CE may be to ED in the given ratio. Join AD, and from E draw EF to meet AC in F, and making the angle CFE less than CAD. Through F draw FG parallel to CB meeting AB in G, join DG and produce it to meet CA produced in H. Then DH is divided in G in the given ratio. What are the limits to the position of the point F?

33. Let P be the given point and AB the given straight line. Draw PQ perpendicular to AB and produce QP to R making QP to PR in the given ratio; through R draw CD parallel to AB: then any line drawn through P and terminated by the parallels will be divided at P in the given ratio.

34. Let AB be the given line from which it is required to cut off a part BC such that BC shall be a mean proportional between the remainder AC and another given line. Produce AB to D, making BD equal to the other given line. On AD describe a semicircle, at B draw BE perpendicular to AD. Bisect BD in O, and with centre O and radius OB describe a semicircle, join OE cutting the semicircle on BD in F, at F draw FC perpendicular to OE and meeting AB in C. C is the point of division, such that BC is a mean proportional between AC and BD.

35. Take any straight line AB, and find another AC, so that AC is to AB as $\sqrt{5} : 1$; and let AB, AC make any acute angle at A and join BC. On AC take AD equal to the given straight line, and through D, draw DE parallel to CB, and meeting AB in E, then AE is the line required.

36. Find two squares in the given ratio, and if BF be the given line (figure Euc. VI. 4), draw BE at right angles to BF, and take BC, CE respectively equal to the sides of the squares which are in the given ratio. Join EF, and draw CA parallel to EF: then BF is divided in A as required.

37. See Euc. VI. 13.

38. This may be effected in different ways, one of which is the following. At one extremity A of the given line AB draw AC making any acute angle with AB and join BC: at any point D in BC draw DEF parallel to AC cutting AB in E and such that EF is equal to ED, draw FC cutting AB in G. Then AB is harmonically divided in E, G.

39. In the fig. Euc. VI. 13. DB is the Geometric mean between AB and BC, and if AC be bisected in E, AE or EC is the Arithmetic mean.

The next the same as—to find the segments of the hypothenuse of a right-angled triangle made by a perpendicular from the right angle, having given the difference between half the hypothenuse and the perpendicular.

40. For "the base produced," read "*the part of the base produced.*" Let ABC be the given right-angled triangle, C the right angle, and BC the base.

At the vertex A in AB, make the angle BAD equal to the angle BAC, and let AD meet the base CB produced in D. Then D is the point required. Euc. VI. 3.

41. The construction is suggested by Euc. I. 47, and Euc. VI. 31.

43. Produce one side of the triangle through the vertex, and make the part produced equal to the other side. Bisect this line, and with the vertex of the triangle as centre and radius equal to half the sum of the sides, describe a circle cutting the base of the triangle.

44. This Problem is analogous to Problem 24, p. 349.

45. Suppose that ABCD the required square is constructed; and PA, PB, PC, the distances of the point P from the three angles of the square are given. Draw BQ perpendicular to PB and equal to it, and join QC. Then since ABC, PDQ are right angles, the angles ABP, CBQ are equal, and hence QC equal to AP is given, and P, Q are given points. Wherefore PQ being given in magnitude and position, and QC, CP in magnitude; the point C is given, and hence the side BC of the square.

Construction. Draw BQ perpendicular and equal to PB, the line which lies between AP, CP, and join PQ: on PQ as a base, and PA, PQ as sides, describe the triangle PCQ: then BC is the side of the square.

46. Let AB, AC, be the two given lines placed at right angles at A. Take AC

to AD in the given ratio, and join CD; with centre B and radius BP equal to the side of the given square, describe a circle cutting CD in P; draw PE parallel to AC, and EF parallel to CD: then AB, AC are divided in E and F as required.

47. If a triangle be constructed on AB so that the vertical angle is bisected by the line drawn to the point C. By Euc. VI. A. the point required may be determined.

48. Let AB, AC be the two given straight lines meeting at A, and P the given point between them. At A draw AD, AE so that AD is to AE in the given ratio, and containing the angle ADE equal to the given angle. Join DE, PA, through P draw PF parallel to AD meeting AB in F, and PG parallel to AE meeting AC in G. PF, PG, are the lines required.

49. Let the given line AB be divided in C, D. On AD describe a semicircle, and on CB describe another semicircle intersecting the former in P; draw PE perpendicular to AB; then E is the point required.

50. The line drawn through the given point and making equal angles with the two given lines is the line required. If a circle be described touching the two given lines at the points where the required line meets them, the rectangle contained by the segments of any other line drawn through the given point, is greater than the rectangle by the segments of that line which makes equal angles with the given line.

51. This problem is misplaced: it properly falls among the problems on Euc. XI. Let ABCD be the given rectangle, whose sides AB, CD are each equal to $34a$, and whose other sides are each equal to $13a$. Take A$h$, A$k$, B$e$, B$f$ each equal to $12a$, join $ef$, $hk$, these will be the creases required. For draw A$m$, B$g$ perpendicular to $hk$, $ef$, and join $gm$; bisect CD in P; draw PR parallel to BC, and AD meeting $gm$ in Q; join P$g$, P$m$; let the triangles B$fc$, A$hk$ be turned about $fe$ and $hk$ till they take the respective positions $fce$, $hdk$, at right angles to the plane of the rectangle ABCD; and join P$c$, P$d$.

52. Let ABC be a right-angled triangle, having the right angle at B and the base BC greater than the perpendicular AB. Let P be the required point in BC (not AC) so that when AP is joined, and PD drawn perpendicular to AC, AP and PD shall be a minimum. Produce DP to meet AB produced in E. Then ED or AP and PD may be proved to be less than the sum of two lines drawn to any other point of BC.

53. At any point D in BC draw DE perpendicular to BC meeting AB in E; in EA take EF to ED as 1 to 2, and join FD. Through A draw AG parallel to FD, and through G draw GP parallel to DE meeting AB in P. Then P is the point such that AP is half of PG.

54. Let these points be taken, one on each side, and straight lines be drawn to them: it may then be proved that these points severally bisect the sides of the triangle.

55. Let AB be equal to a side of the given square. On AB describe a semicircle; at A draw AC perpendicular to AB and equal to a fourth proportional to AB and the two sides of the given rectangle. Draw CD parallel to AB meeting the circumference in D. Join AD, BD, which are the required lines.

56. Describe a circle about the triangle, and draw the diameter through the vertex A, draw a line touching the circle at A, and meeting the base BC produced in D. Then AD will be a mean proportional between DC and DB. Euc. III. 36.

57. Let A, B be the two given points, and C a point in the circumference of the given circle. Let a circle be described through the points A, B, C and cutting the circle in another point D. Join CD, AB, and produce them to meet in E. Let EF be drawn touching the given circle in F, the circle described through the points A, B, F, will be the circle required. Joining AD and CB, by Euc. III. 21, the triangles CEB, AED are equiangular, and by Euc. VI. 4, 16, III. 36, 37, the given circle and the required circle each touch the line EF in the same point, and therefore touch one another. When does this solution fail?

Various cases will arise according to the relative position of the two points and the circle.

58. Let A be the given point, BC the given straight line, and D the centre of the given circle. Through D draw CD perpendicular to BC, meeting the circumference in E, F. Join AF, and take FG to the diameter FE, as FC is to FA. The circle de-

scribed passing through the two points A, G and touching the line BC in B is the circle required. Let H be the centre of this circle; join HB, and BF cutting the circumference of the given circle in K, and join EK. Then the triangles FBC, FKE being equiangular, by Euc. VI. 4, 16, and the construction, K is proved to be a point in the circumference of the circle passing through the points A, G, B. And if DK, KH be joined, DKH may be proved to be a straight line:—the straight line which joins the centres of two circles, and passes through a common point in their circumferences.

59. Let A be the given point, B, C the centres of the two given circles. Let a line drawn through B, C meet the circumferences of the circles in G, F; E, D, respectively. In GD produced, take the point H, so that BH is to CH as the radius of the circle whose centre is B, is to the radius of the circle whose centre is C. Join AH, and take KH to DH as GH to AH. Through A, K describe a circle ALK touching the circle whose centre is B, in L. Then M may be proved to be a point in the circumference of the circle whose centre is C. For by joining HL and producing it to meet the circumference of the circle whose centre is B in N; and joining BN, BL, and drawing CO parallel to BL, and CM parallel to BN, the line HN is proved to cut the circumference of the circle whose centre is B in M, O; and CO, CM are radii. By joining GL, DM, M may be proved to be a point in the circumference of the circle ALK. And by producing BL, CM to meet in P, P is proved to be the centre of ALK, and BP joining the centres of the two circles passes through L the point of contact. Hence also is shewn that PMC passes through M, the point where the circles whose centres are P and C touch each other.

NOTE. If the given point be in the circumference of one of the circles, the construction may be more simply effected thus:

Let A be in the circumference of the circle whose centre is B. Join BA, and in AB produced, if necessary, take AD equal to the radius of the circle whose centre is C, join DC, and at C make the angle DCE equal to the angle CDE, the point E determined by the intersection of DA produced and CE, is the centre of the circle.

60. Let the two given circles be without one another, and let A, B be their centres. Join AB cutting the circumferences in C, D; take CE, DF each equal to the radius of the required circle: the two circles described with centres A, B, and radii AE, BF respectively, will intersect one another, and the point of intersection will be the centre of the required circle. Distinguish the different cases.

61. Let the two given lines AB, BD meet in B, and let C be the centre of the given circle, and let the required circle touch the line AB, and have its centre in BD. Draw CFE perpendicular to HB intersecting the circumference of the given circle in F, and produce CE, making EF equal to the radius CF. Through G draw GK parallel to AB, and meeting DB in K. Join CK, and through B, draw BL parallel to KC, meeting the circumference of the circle whose centre is C in L; join CL and produce CL to meet BD in O. Then O is the centre of the circle required. Draw OM perpendicular to AB, and produce EC to meet BD in N. Then by the similar triangles, OL may be proved equal to OM.

62. Let AB be the given straight line, and C the centre of the given circle; through C draw the diameter DCE perpendicular to AB. Place in the circle a line FG which has to AB the given ratio; bisect FG in H, join CH, and on the diameter DCE, take CK, CL, each equal to CH; either of the lines drawn through K, L, and parallel to AB is the line required.

63. The locus of the intersections of the diagonals of all the rectangles inscribed in a scalene triangle, is a straight line drawn from the bisection of the base to the bisection of the shorter side of the triangle.

64. The tangent AC by Euc. III. 36, may be shewn to be equal to 3 . AB: the problem is reduced to finding a mean proportional between AB and 3 . AB.

65. Let A be the given point within the circle whose centre is C, and let BAD be the line required, so that BA is to AD in the given ratio. Join AC and produce it to meet the circumference in E, F. Then EF is a diameter. Draw BG, DH perpendicular on EF: then the triangles BGA, DHA are equiangular. Hence the construction.

66. This Problem only differs from the preceding in having the given point without the given circle.

67. Draw tangents EF, GH, common to the two circles whose centres are A, B to cut the line AB in C and D, (C being between A and B, and D in AB produced): then any line drawn through either of these points will fulfil the conditions of the Problem.

Take a line through D, the intercepted chords of which are KL and MN; and join AM, AN, AE, BK, BL, BF. Then AD : DB :: AE : BF :: AN : BL.

Whence the triangles ADN, BDL are similar, and AN is parallel to BL. Similarly AM is parallel to BK; and hence the isosceles triangles MAN, KBL have the angles at their vertices A, B, equal. The other angles therefore are equal, and the triangles are similar. Whence MN : KL :: AM : BK.

Hence to draw a pair of common tangents to two circles.

Draw any two parallel radii AM, BK, join MK, AB and produce them to intersect in D. Then D is the point from which a pair of common tangents may be drawn to the two circles. The points of contact may be found by describing circles on BD, DA as diameters.

68. Since the magnitude and position of the two circles are given, their radii and the distances of their centres are known. Let O, O′ be the centres of the larger and smaller circles, join OO′, O′C, from C draw CE to touch the larger circle in E. Join also OE, CO. Then Euc. III. 36. CA . CB = CE², and from the right-angled triangles CE is known, also the ratio of CA to CB is given. Hence the problem is reduced to this: to find two lines which have a given ratio to one another, and whose rectangle is equal to a given rectangle.

69. Let A be the given point in the circumference of the circle, C its centre. Draw the diameter ACB, and produce AB to D, taking AB to BD in the given ratio: from D draw a line to touch the circle in E, which is the point required. From A draw AF perpendicular to DE, and cutting the circle in G.

70. See Problem 8, p. 348.

71. Suppose the area of the grass-plot to be two-thirds of the rectangular field.

Let HM, LM be two lines drawn from H, L points in AB, BC and parallel to the sides, cut off the rectangle HMLB equal to two-thirds of ABCD, Problem 95, p. 352. Bisect LC, AH in P, Q, and through P, Q draw PR, QR parallel to the sides and meeting in R. On BA, BC, take BS, BT each equal to AQ, through S draw SX parallel to BC and meeting QR in X, and through T draw TYZ parallel to AB meeting SX in Y, and QR in Z. Then the rectangle RXYZ is equal to HMLB, and AQ is the width of the pathway round the grass-plot.

72. By means of Theorem 137, p. 362, the ratio of the diagonals AC to BD may be found to be as AB . AD + BC . CD to AB . BE + AD . DC. figure, Euc. VI. D.

73. The two hexagons consist each of six equilateral triangles, and the ratio of the hexagons is the same as the ratio of their equilateral triangles.

74. Let ABC be any triangle and D be the given point in BC, from which lines are to be drawn which shall divide the triangle into any number (suppose five) equal parts. Divide BC into five equal parts in E, F, G, H, and draw AE, AF, AG, AH, AD, and through E, F, G, H draw EL, FM, GN, HO parallel to AD, and join DL, DM, DN, DO; these lines divide the triangle into five equal parts.

By a similar process, a triangle may be divided into any number of parts which have a given ratio to one another.

75. Let ABC be the larger, *abc* the smaller triangle, it is required to draw a line DE parallel to AC cutting off the triangle DBE equal to the triangle *abc*. On BC take BG equal to *bc*, and on BG describe the triangle BGH equal to the triangle *abc*. Draw HK parallel to BC, join KG; then the triangle BGK is equal to the triangle *abc*. On BA, BC take BD to BE in the ratio of BA to BC, and such that the rectangle contained by BD, BE shall be equal to the rectangle contained by BK, BG. Join DE, then DE is parallel to AC, and the triangle BDE is equal to *abc*.

76. Let ABC be the given triangle which is to be divided into two parts having a given ratio, by a line parallel to BC. Describe a semicircle on AB and divide AB in D in the given ratio, at D draw DE perpendicular to AB and meeting the circumference in E, with centre A and radius AE describe a circle cutting AB in F; the line drawn through F parallel to BC is the line required.

In the same manner a triangle may be divided into three or more parts having any given ratio to one another by lines drawn parallel to one of the sides of the triangle.

77. This is a particular case of Prob. 1, p. 254. It will be found that the side of the square is equal to one-third of the hypothenuse.

78. The square inscribed in a right-angled triangle which has one of its sides coinciding with the hypothenuse, may be shewn to be less than that which has two of its sides coinciding with the base and perpendicular.

79. Find M and N the sides of squares equal to the given triangle and the given area of the rectangle respectively; from A draw AD perpendicular on BC the base of the triangle; on AD describe a semicircle, bisect the arc in E and draw ED; next find a fourth proportional to M, N, and DE. In DB make DF equal to this line, and draw FGG′ parallel to AD and cutting the semicircle in G, G′: then one side of the required rectangle passes through G.

A second rectangle fulfilling the conditions is found by means of G′.

The limit of possibility is when FGG′ becomes a tangent to the semicircle; and the inscribed rectangle is then the greatest possible.

Give the Analysis of the Problem.

80. Take two points on the radii equidistant from the centre, and on the line joining these points, describe a square; the lines drawn from the centre through the opposite angles of the square to meet the circular arc, will determine two points of the square inscribed in the sector.

81. Since the area of the triangle is given, the side of the square which is double the area of the triangle is given. Find a third proportional to the given base, and the side of this square: this line will be equal to the altitude of the triangle.

82. The sides of the three squares inscribed in the triangle may be shewn to be inversely as the three sides of the triangle respectively assumed as the base. Hence that square is the greatest which is contiguous to the smallest of the three sides of the triangle.

83. Let ABCDE be the given regular pentagon. On AB, AE take equal distances AF, AG, join FG, and on FG describe a square FGKH. Join AH and produce it to meet a side of the pentagon in L. Draw LM parallel to FH meeting AE in M. Then LM is a side of the inscribed square.

84. Let ABC be the given triangle. Draw AD making with the base BC an angle equal to one of the given angles of the parallelogram. Draw AE parallel to BC and take AD to AE in the given ratio of the sides. Join BE cutting AC in F.

85. The greatest parallelogram which can be inscribed in a given triangle, is that which has one side parallel to the base of the triangle, and terminated in the points of bisection of the sides.

86. This is only a particular case of the general problem, when the rectilineal figures are a right-angled triangle and a square of given magnitudes.

87. Let AB be the base of the segment ABD fig. Euc. III. 30. Bisect AB in C, take any point E in AC and make CF equal to CE: upon EF describe a square EFGH: from C draw CG and produce it to meet the arc of the segment in K.

88. This parallelogram may be proved to be a square.

89. Analysis. Let ABCD be the given rectangle, and EFGH that required, having the point E on AB, F on BC, &c., and let LMKN be the parallelogram whose angular points, L, M, N, K are at the bisections of AB, BC, CD, DA.

Then $HA.AE+EB.BF=$ fig. ABCD $-$ fig. EFGH $=$ a given area.

Whence $2.HK.EL=$ fig. ABCD $-$ fig. EFGH $-2.AK.AL$.

The rectangle of the two distances HK, EL, from the middle of the sides is therefore given.

Also by the similar triangles HAE, EBF, may be deduced,

$$LE^2-HK^2=AL^2-AK^2.$$

Whence the rectangle, and the difference of the squares of LE, KH are given, to find the lines themselves.

90. In a straight line at any point A, make $Ac$ to $Ad$ in the given ratio. At A draw AB perpendicular to $cAd$, and equal to a side of the given square. On $cd$ describe a semicircle cutting AB in $b$; and join $bc$, $bd$; from B draw BC parallel to $bc$, and BD parallel to $bd$: then AC, AD are the adjacent sides of the rectangle. For, CA is to AD as $cA$ to $Ad$; Euc. VI. 2. and $CA.AD=AB^2$, CBD being a right-angled triangle.

91. Let D be the given point *within* the triangle ABC. In the base BC take BX to BC as the part to be cut off is to the whole triangle. Join BX, and describe

the parallelogram BEFG equal to the triangle ABX, having the sides BE, BG on BA, BC respectively, and the side EF passing through the point D. Draw DH parallel to AB, and on HG describe a semicircle, place HK in this semicircle, equal to HB, join GK, and with centre G and radius GK describe a circle cutting BC in L, M; the line drawn from L or M through D, cuts off the required part from the triangle ABC. Give the analysis.

92. A mean proportional between two homologous sides of the polygons, will be the corresponding side of the required polygon.

93. Analysis. Let ABCD be the given rectangle, and EFGH that to be constructed. Then the diagonals of EFGH are equal and bisect each other in P the centre of the given rectangle. About EPF describe a circle meeting BD in K, and join KE, KF. Then since the rectangle EFGH is given in species, the angle EPF formed by its diagonals is given; and hence also the opposite angle EKF of the inscribed quadrilateral PEKF is given. Also since KP bisects that angle, the angle PKE is given, and its supplement BKE is given. And in the same way, KF is parallel to another given line; and hence EF is parallel to a third given line. Again, the angle EPF of the isosceles triangle EPF is given; and hence the quadrilateral EPFK is given in species.

94. Reference may be made to Euc. VI. 31.

95. It is manifest that this is the general case of Prob. 3, p. 331.

If the rectangle to be cut off be two-thirds of the given rectangle ABCD.

Produce CB to E so that BE may be equal to a side of that square which is equal to the rectangle required to be cut off; in this case, equal to two thirds of the rectangle ABCD. On AB take AF equal to AD or BC; bisect FB in G, and with centre G and radius GE describe a semicircle meeting AB, and AB produced, in H and K. On CB take CL equal to AH and draw HM, LM parallel to the sides, and HBLM is two-thirds of the rectangle ABCD.

96. Draw any diameter AB, in AB take any point C, and through C draw DCE perpendicular to AB, making CE and CD each equal to half of AC. Join AD, AE and produce them to meet the circumference in F, G. Join FG.

97. Draw any diameter AB of the given circle; on AB take AC to AB as 4 to 5, and draw DCE perpendicular to AB. Join AE, EB, BD, BA, and the isosceles triangle AED is four times the isosceles triangle EBD.

98. In the figure Euc. III. 30; from C draw CE, CF making with CD, the angles DCE, DCF each equal to the angle CDA or CDB, and meeting the arc ADB in E and F. Join EF, the segment of the circle described upon EF and which passes through C will be similar to the segment ADB.

99. The side of the equilateral triangle is one-third, and the side of the regular pentagon is one-fifth of the given line. The *radii* of their inscribed circles may be expressed in terms of the sides of the triangle and pentagon respectively: and the numerical ratio of the radii will be found to be as $5\sqrt{5-\sqrt{5}}$ to $\sqrt{21}$.

100. By means of the similar triangles it may be proved that of any three of the consecutive circles, the sum of the radii of the first and second, is to their difference, as the sum of the radii of the second and third, is to their difference.

## HINTS, &c. TO THE THEOREMS. BOOK VI.

3. This is the converse of the Cor. Euc. VI. 1, and may be proved indirectly.

4. See Note p. 5, Appendix.

5. See Note Euc. VI. A, p. 203.

6. The lines drawn making equal angles with homologous sides, divide the triangles into two corresponding pairs of equiangular triangles; by Euc. VI. 4, the proportions are evident.

7. Let DE be the position of the given line and B the given point through which the parallel is to pass. Stretch the string from B to D and take a continued length DA equal to DB. From A take any length to meet DE in any point E, and take a continued length EC equal to EA: the line joining B and C is parallel to DE.

8. All the parallelograms are manifestly similar and similarly situated with respect to each other: and every 4, 9, 16, 20, &c. of the smaller ones, form parallelograms similar to each other and the smaller one.

9. Join BC, DC. Draw AF intersecting DC in G, and let AF produced meet BC in H. Then DC is parallel to BC, and from the similar triangles FDG, FHC; FDE, FBC; ADE, ABC; FC may be shewn to be equal to $m$. FD: and DC = FC + DF.

10. Join $ba$ cutting AE in F. Then by the similar triangles D$b$F, DFC; D$ba$, DBC; A$ba$, ABC; $n$. FE = $(n+1)$. DE, and DE = AE − AD. Also from the similar triangles A$ba$, ABC; $n$. FE = $(n-1)$. AE. Whence is shewn 2. AE = $(n+1)$. AD. By means of the Theorem 160, p. 364, and Euc. VI. 2, (for $ba$ is parallel to BC) BC may be proved to be bisected in E.

11. By constructing the figure, the angles of the two triangles may easily be shewn to be respectively equal.

12. The area of a right-angled triangle being half of the rectangle contained by the base and perpendicular; the sum of the series of triangles will be found to depend upon the sum of the series $1 + \frac{1}{2} + \frac{1}{4} +$ &c. continued ad infinitum.

13. This may be shewn from Euc. I. 47; VI. 4, 16.

14. Let ABCD be a square and AC its diagonal. On AC take AE equal to the side BC or AB: join BE and at E draw EF perpendicular to AC and meeting BC in F. Then EC, the difference between the diagonal AC and the side AB of the square, is less than AB; and CE, EF, FB may be proved to be equal to one another: also CE, EF are the adjacent sides of a square whose diagonal is FC. On FC take FG equal to CE and join EG. Then as in the first square, the difference CG between the diagonal FC and the side EC or EF, is less than the side EC. Hence EC the difference between the diagonal and the side of the given square, is contained twice in the side BC with a remainder CG: and CG is the difference between the side CE and the diagonal CF of another square. By proceeding in a similar way, CG the difference between the diagonal CF and the side CE, is contained twice in the side CE with a remainder: and the same relations may be shewn to exist between the difference of the diagonal and the side of every square of the series which is so constructed. Hence, therefore, as the difference of the side and diagonal of every square of the series, is contained twice in the side with a remainder, it follows that there is no line which exactly measures the side and the diagonal of a square.

15. In the arc AB (fig. Euc. IV. 2) let any point K be taken, and from K let KL, KM, KN be drawn perpendicular to AB, AC, BC respectively, produced if necessary, also let LM, LN be joined, then MLN may be shewn to be a straight line. Draw AK, BK, CK, and by Euc. III. 31, 22, 21; Euc. I. 14.

16. Join AL, and produce it to meet the base BC in G. Join also DF intersecting AG in M. Then every line DF drawn parallel to the base BC is divided in the same proportion as the segments of the base made by a line drawn from the vertex of the triangle. Then conversely, if this proportion hold good, the line joining the points L, G must pass through A.

17. Divide the given base BC in D, so that BD may be to DC in the ratio of the sides. At B, D draw BB′, DD′ perpendicular to BC and equal to BD, DC respectively. Join B′D′ and produce it to meet BC produced in O. With centre O and radius OD describe a circle. From A any point in the circumference join AB, AC, AO. Prove that AB is to AC as BD to DC. Or thus. If ABC be one of the triangles. Divide the base BC in D so that BA is to AC as BD to DC. Produce BC and take DO to OC as BA to AC: then O is the centre of the circle.

18. A circle may be described about the four-sided figure ABDC. By Euc. I. 13; Euc. III. 21, 22. The triangles ABC, ACE may be shewn to be equiangular.

19. Let ABC be the given triangle, and let the line EGF cut the base BC in G. Join AG. Then by Euc. VI. 1, and Theo. 85, p. 358, it may be proved that AC is to AB as GE is to GF.

20. Since CE is equal to CA, the triangles CAB, CED are similar, Euc. VI. 6, and by Euc. I. 5, 32, the triangles CAB, DCB may be shewn to be similar.

21. Let ABC be the triangle, right-angled at C, and let AE on AB be equal to AC, also let the line bisecting the angle A, meet BC in D. Join DE. Then the triangles ACD, AED are equal, and the triangles ACB, DEB equiangular.

22. Let ABC be any triangle, let BD be drawn parallel to AC and equal to AB, and CE parallel to AB and equal to AC. Join DC, BE intersecting AB, AC in F, G respectively. Then by means of the similar triangles, two proportions may be found from which it may be proved that AF is equal to AG, and that either is a mean proportional between BF and CG.

23. See Theorem 16, p. 308.

24. If the property be assumed to be true; then by Euc. VI. 16, it follows that the difference of the squares of the sides of the triangle, is equal to the difference of the squares of the segments of the base; and therefore the difference of the squares of one side and the adjacent segment of the base, is equal to the difference of the squares of the other side and its adjacent segment of the base. Whence it follows, that the line drawn from the vertex of the triangle to the point of section of the base, is perpendicular to the base.

25. Let BCDE be the square on the side BC of the isosceles triangle ABC. Then by Euc. VI. 2, FG is proved parallel to ED or BC.

26. Let the $n^{th}$ part of the given line be cut off by Euc. VI. 9, then by Euc. II. 1.

27. The case in which the two angles are equal is proved in Euc. VI. 15, the case where one angle is the supplement of the other offers no difficulty.

28. Produce EG, FG to meet the perpendiculars CE, BF, produced, if necessary; the demonstration is obvious.

29. Let the perpendiculars from B, C the angles at the base, meet the line bisecting the vertical angle A in E, F; and let the line bisecting the vertical angle, meet the base in D. Then twice the area of the triangle ABC is the sum of the rectangles contained by AD, BE and AD, CF. The triangles AFC, AEB are equiangular, as also the triangles CFD, BED.

30 This property is a particular case of Euc. VI. 2.

31. The lines joining the bisections of every two sides may be proved parallel to the remaining side of the triangle, and the equality of the triangles may be inferred from Euc. I. 38.

32. Draw AF, and the triangle AFC is equal to the triangle ABD; therefore the ratio of the triangle FCE to ABD is known. The numerical value of the ratio may be found from the note on Euc. II. 11, p. 72.

The second property is obvious from the similarity of the triangles.

33. This property may be deduced directly from Euc. VI. B, 3.

34. This property may be immediately deduced from Euc. VI. 8, Cor.

35. From D draw DE perpendicular to AB, then DE is equal to DC.

And by Euc. VI. 3, 4, $DC : AC :: BE : BC$.

Whence may be shewn, $AC^2 : AD^2 :: BC^2 : BE^2 + BC^2$:

also $BE^2 = BD^2 - DE^2 = BD^2 - DC^2 = (BD + DC).(BD - DC) = BC.(BD - DC)$.

Whence it follows that $AC^2 : AD^2 :: BC^2 : BC\,(BD - DC + BC) :: BC : 2.BD$.

36. Let the line DF drawn from D the bisection of the base of the triangle ABC, meet AB in E, and CA produced in F. Also let AG drawn parallel to BC from the vertex A, meet DF in G. Then by means of the similar triangles; DF, FE, FG may be shewn to be in harmonic proportion.

37. See Euc. VI. A, note, p. 204.

38. Produce AC to G. Then the angle BCG is bisected by CF, Euc. VI. A, and the angle ACB by CE. Euc. VI. 3.

39. The angle at A the centre of the circle (fig. Euc. IV. 10) is one tenth of four right angles, the arc BD is therefore one tenth of the circumference, and the chord BD is the side of a regular decagon inscribed in the larger circle. Produce BC to meet the circumference in F and join BF, then BF is the side of the inscribed pentagon, and AB is the side of the inscribed hexagon. Join FA. Then FCA may be proved to be an isosceles triangle and FB is a line drawn from the vertex meeting the base produced. If a perpendicular be drawn from F on BC, the difference of the squares of FB, FC may be shewn equal to the rectangle AB, BC, (Euc. I. 47; II. 5, Cor.); or the square of AC, Euc. IV. 10.

40. The triangles formed by drawing the successive perpendiculars may be shewn to be equiangular, and each equiangular to the original triangle.

41. Draw the perpendicular CE from C on the base AB.

Then $CB^2 = CD^2 + BD^2 + 2 . BD . DE$. Euc. II. 12.

and $AC^2 = CD^2 + AD^2 - 2 . AD . DE$. Euc. II. 13.

Multiply the former by AD, and the latter by BD, and add the results,

$$AD . BC^2 + BD . AC^2 = CD^2 (AD + BD) + AD . BD (BD + AD)$$
$$= CD^2 . AB + AD . BD . AB.$$

This result is Prop. II. of Matthew Stewart's General Theorems.

The Analytical Discussion of Matthew Stewart's General Theorems by T. S. Davies, Esq., F.R.S., will be found at p. 573, &c. Vol. XV. of the Transactions of the Royal Society of Edinburgh.

42. The segments cut off from the sides are to be measured from the right angle, and by similar triangles are proved to be equal; also by similar triangles, either of them is proved to be a mean proportional between the remaining segments of the two sides.

43. The triangles HCF, ABF may be shewn to be equiangular.

44. Draw FG perpendicular to BA, and FH perpendicular to CE. From the similar triangles AED, AGF, BFG, BCE; DE : CE :: BG : AG.

But $BG = AE - EG = AE - FH$, and $AG = AE + EG = AE + FH$. And from the similar triangles CFH, CEA, AC : CF :: AE : FH. Whence may be deduced the proportion DE : EC :: AC − CF : AC + CF.

45. This theorem is the same as theorem 19, p. 354, under a slightly modified form of expression.

46. Assuming the truth of Theorem 35, page 355, namely,

$$AC^2 : AD^2 :: BC : 2 . BD; \quad \therefore\ 2 . AC^2 : AD^2 :: BC : BD,$$

whence $2 . AC^2 - AD^2 : AD^2 :: BC - BD : BD$,

$$\text{and since } 2 . AC^2 - AD^2 = 2 . AC^2 - (AC^2 + DC^2) = AC^2 - CD^2,$$

the property is immediately deduced.

47. The enunciation of this theorem does not seem definite.

48. See Theorem 2, page 305.

49. Each of the lines may be proved to be divided at the point of intersection in the ratio of 2 to 1.

50. This theorem bears the same relation to Euc. VI. B as Euc. VI. A does to Euc. VI. 3. Describe a circle about the triangle ABC, produce EA to meet the circumference again in F and join FC. Then by the similar triangles BEA, FCA; the rectangle BA, AC is equal to the rectangle AE, AF. By Euc. III. 36, Cor. the rectangle FE, EA is equal to the rectangle BE, EC. And since FE is divided in A, Euc. II. 3, the rectangle FE, EA is equal to the rectangle EA, AF together with the square of AE. Hence, &c.

51. In the figure, Theorem 45, p. 302, draw PQ, PR, PS perpendiculars on AB, AD, AC respectively: then since the triangle PAC is equal to the two triangles PAB, PAD, it follows that the rectangle contained by PS, AC, is equal to the *sum* of the rectangles contained by PQ, AB, and by PR, AD. When is the rectangle by PS, AC, equal to the *difference* of the other two rectangles?

52. Suppose $Bfh$ to intersect EF, EG, EH in $f$, $g$, $h$, and to meet EK in K. Through $h$ draw $hkl$ parallel to EB or AC and $km$ parallel to EF. Then by means of the similar triangles may be proved, that $Bf$ is to $fh$ as $fg$ is to $2mg$. Whence $Bf$ is to $Bh$ as $fg$ is to $gh$, since $2mg$ is the difference between $fg$ and $gh$.

Again, draw $Hn$ parallel to $Gg$, and by a similar process is proved, $Bg$ is to BK as $gh$ to $h$K. The line $Bfh$ might be drawn so as to meet any one of the equidistant points in the given line AC.

53. The angles made by the four lines at the point of their divergence remain constant. See Note on Euc. VI. A, p. 203.

54. The triangles AEC, CBE may be shewn to be equiangular. Then Euc. VI. 4.

55. This is an extension of theorem 49, page 356. If the base BC of the triangle ABC be produced to E, so that CE is equal to BC, and AC be bisected in D, then if BD be joined and produced to meet AE in F, then AF is one half of FE.

56. If the vertex of the triangle be in one of the sides, the inscribed square is

greatest when its altitude is greatest, or when the base of the triangle coincides with the base of the square. Every other inscribed triangle may be shewn to be less than this triangle.

57. Let ABCDEF be any hexagon inscribed in a circle, and let the opposite sides AB, DE; BC, FE; CD, FA when produced meet in P, Q, R respectively. Produce AB, DC to meet in $a$, and EF both ways to meet BAP in $b$ and CDR in $c$. Then by Euc. III. 36, and considering $abc$ as a triangle intersected by the transversals PED, QCB, RFA respectively: it may be proved that $a\mathrm{P}.b\mathrm{Q}.c\mathrm{R}=\mathrm{P}b.\mathrm{Q}c.\mathrm{R}a$, which is the condition fulfilled when a straight line intersects the three sides $ab$, $ac$, $bc$ of the triangle produced, in the points P, Q, R. See Appendix, p. 20.

58. The hexagon is not necessarily regular. By joining the points of contact, the three diagonals may be proved to intersect in one point by means of poles and polars.

59. The square inscribed in the circle may be shewn to be equal to twice the square of the radius, and five times the square inscribed in the semicircle, to four times the square of the radius.

60. The three triangles formed by three sides of the square with segments of the sides of the given triangle, may be proved to be similar. Whence by Euc. VI. 4, the truth of the property.

61. By constructing the figure, it may be shewn that twice the square inscribed in the quadrant is equal to the square of the radius, and that five times the square inscribed in the semicircle is equal to four times the square of the radius. Whence it follows that, &c.

62. By Euc. I. 47, and Euc. VI. 4, it may be shewn that four times the square of the radius is equal to fifteen times the square of one of the equal sides of the triangle.

63. See Problem 30, page 334.

64. The triangle cut off may be proved to be one-fourth of the given triangle.

65. In the figure Euc. IV. 7. Draw AD, then AD is the side of the inscribed square. Join EF, meeting AD in L, and the circumference in M, and draw AM; AM is the side of the inscribed octagon. Then it may be shewn that $\mathrm{A}_1:\mathrm{A}::\mathrm{EL}:\mathrm{EA}$, and $\mathrm{A}:\mathrm{A}_2::\mathrm{EA}:\mathrm{EF}$; also since the triangles ELA, EFA are similar, $\mathrm{EL}:\mathrm{EA}::\mathrm{EA}:\mathrm{EF}$. Whence it follows that, &c.

66. The intersection of the diagonals is the common vertex of two triangles which have the parallel sides of the trapezium for their bases.

67. From one of the given points two straight lines are to be drawn perpendicular, one to each of any two adjacent sides of the parallelogram; and from the other point, two lines perpendicular in the same manner to each of the two remaining sides. When these four lines are drawn to intersect one another, the figure so formed may be shewn to be equiangular to the given parallelogram.

68. Suppose the side AD greater than BC, and from B, C, draw BE, CE' perpendiculars on AD. Then by the right-angled triangles, &c.

69. Let AK be any rectangle contained by two lines AB, BC, and CK, AE the squares upon BC, AE. Then by Euc. VI. 1.

70. This is only another form of stating the general property of three lines in harmonical proportion. It may be deduced from the note to Euc. VI. A, page 203.

71. There appears to be some inaccuracy in the enunciation: for four lines PA, PB, PC, PD in harmonical proportion cannot be drawn from a point P to meet a straight line in four points A, B, C, D taken in order.

72. See Appendix, p. 20.

73. Let the figure be drawn, and let HP, KQ, LR, when produced, meet in O: also let KH, PQ meet in M; HL, PR in N; KL, QR in S. Then, since the triangles HKO, KLO, HLO, are cut by the transversals, PQM, QRS, PRN respectively; therefore

$$\frac{\mathrm{KQ}}{\mathrm{QO}}\cdot\frac{\mathrm{OP}}{\mathrm{PH}}\cdot\frac{\mathrm{HM}}{\mathrm{MK}}=1;\quad \frac{\mathrm{KQ}}{\mathrm{QO}}\cdot\frac{\mathrm{OR}}{\mathrm{RL}}\cdot\frac{\mathrm{LS}}{\mathrm{SK}}=1;\quad \frac{\mathrm{LR}}{\mathrm{RO}}\cdot\frac{\mathrm{OP}}{\mathrm{PH}}\cdot\frac{\mathrm{HN}}{\mathrm{NL}}=1;$$

whence is deduced $\frac{\mathrm{HM}}{\mathrm{MK}}\cdot\frac{\mathrm{KS}}{\mathrm{SL}}\cdot\frac{\mathrm{LN}}{\mathrm{NH}}=1$; which is the condition fulfilled when the three sides of the triangle HKL are produced, and are cut by a transversal in the points M, N, S. Hence the points M, N, S are in a straight line.

74. Let a point $m$ be taken in CD, and $mpq$ be drawn parallel to AB, and

intersecting FE, HG, in $p$, $q$: through $q$, let $rqs$ be drawn parallel to CD, and intersecting EF, AB in $r$, $s$; then $mp : pq :: sq : qr$, may be proved from the similar triangles. In the same manner, if through $s$, $tsv$ be drawn parallel to EF, and intersecting GH, CD in $t$, $v$, then $qs : qr :: ts : sv$, and similarly if a line be drawn through $v$ parallel to the line next in order, &c.

75. See note to Euc. VI. A, page 203.

76. Let ABC be the triangle, M the middle point of BC, and AD = 2 . DM; draw AK, PL parallel to the base, the former meeting EF produced in K, and the latter through D meeting AB and AC in P and L. Then PD = PL, and by the similar triangles KE . FD = KF . ED, or DF . (KD − DE) = (KD + DF) . DE; whence is deduced the equality required.

77. This is the same property as that enunciated in Theorem 53, page 356, in a slightly altered form of expression.

78. Let AB, CD intersect each other in E, and be terminated by two unlimited lines given in position: and let $ab$, $cd$ be drawn parallel to AB, CD respectively, intersecting each other in $e$, and also terminated by the two given lines. Then by the similar triangles and the composition of the ratios.

79. Let PA, PB, PC, PD be four straight lines drawn from P, and let $mpq$ be drawn parallel to PA and meeting PB, PC, PD in $m$, $p$, $q$, so as to be bisected by PC in $p$. Through $p$ draw any line EF$p$G meeting the other three lines in E, F, G. Then EG is divided harmonically in F, $p$.

80. By converting the proportion by Euc. VI. 16, and observing that

$$DB = DC + CB,\quad CB = AC + 2 . CE \text{ and } AD = DC - AC.$$

81. Constructing the figure, the right-angled triangles SCT, ACB may be proved to have a certain ratio, and the triangles ACB, CPM in the same way, may be proved to have the same ratio.

82. Draw DG perpendicular on AE. The triangle CDB is isosceles and DF is drawn from the vertex perpendicular on the base: also the triangles DFB, DAB are equal in all respects. Hence CF, FB, AB, are equal to one another, and AB is half of BC. Similarly BE is half of AB. Then from the similar triangles AGD, FDB, the property may be deduced.

83. See Note Euc. VI. A, p. 203. The bases of the triangles CBD, ACD, ABC, CDE may be shewn to be respectively equal to DB, 2.BD, 3.BD, 4.BD.

84. The triangles DOE, EOB are readily proved to have the same ratio as the triangles EOB, BOA by Euc. VI. 1.

85. This property follows as a corollary to Euc. VI. 23, for the two triangles are respectively the halves of the parallelograms, and are therefore in the ratio compounded of the ratios of the sides which contain the same or equal angles: and this ratio is the same as the ratio of the rectangles by the sides.

86. Let the figure be constructed, then from the three similar right-angled triangles, Euc. VI. 19.

87. Since the three rectangles are equal, AB . AG = CD . CH = EF . EK. Hence AB : CD :: CH : AG and CD : EF :: EK : CH. Then supposing EK − CH = CH − AG, there may be deduced AB − CD : CD − EF :: AB : EF. And conversely. See Note on Def. III. and Prop. A, p. 203.

88. Every triangle may be shewn to be four times the area of the triangle about which it is described.

89. Let C, C′ be the centres of the two circles, and let CC′ the line joining the centres intersect the common tangent PP′ in T. Let the line joining the centres cut the circles in Q, Q′, and let PQ, P′Q′ be joined: then PQ is parallel to P′Q′. Join CP, C′P′, and then the angle QPT may be proved to be equal to the alternate angle Q′P′T.

90. Let C, C′, C″ be the centres of the three circles; C is the centre of the largest, C″ of the smallest. Let the tangents to the circles whose centres are C, C′; C, C″; C′, C″ meet in A, B, C respectively. Join the points A, B, C; then AB shall be in the same straight line as BC. Join C, C′, C″ and produce CC′, CC″, C′C″ these lines meet the tangents in A, B, C respectively. Through C′ draw C′E parallel to AB, then BC may be proved also parallel to C′E.

91. Let the chord AB be bisected in E by the chord CD. Let the tangents at A, B meet in P, and the tangents at C, D meet in Q, join PQ, and PQ is parallel

to AB. Join PE and produce it to the centre O, also join OQ cutting CD in F. Draw the radii OC, OA. Then the triangles OFE, OPQ are equiangular and right-angled, also the right angle BEP is equal to the alternate angle EPQ.

92. Let two circles whose centres are C, C′ cut one another. Let any two points P, Q, be taken in the circumference of one, and tangents be drawn at P, Q. Take P′, Q′ points in the circumference of the other, such that the tangents at P′, Q′ may be parallel to the tangents at P, Q. Draw PP′ intersecting CC′ in D. Join QD, Q′D. If QDQ′ can be proved to be a straight line, then QQ′, and PP′ intersect CC′ in the same point.

Join PC, P′C′, and by the similar triangles PCD, P′C′D the other property is deduced.

93. Join AB, and divide it in C so that AC is to BC as AP to BP: and if in AB produced, ED be taken to BD, as AP to BP; the point D may be proved to be the centre of the circle which is the locus of the point of intersection of the two lines. If the lines be equal, the locus is a straight line. Give the analysis of both cases.

94. The arithmetical ratio of $r \,.\, r'$ to $\frac{1}{2} \,.\, a^2$ may be deduced from Theorem 137, p. 362, Euc. IV. 4, and note to Euc. II. 11, p. 72.

95. This Theorem is the same as Theorem 123, p. 361.

96. By means of Euc. IV. 4, and Theorem 137, p. 362, this Theorem may be shewn to be true.

97. This is the same as Theorem 103, p. 360, under a slightly varied form of expression.

98. Let a tangent be drawn to touch the circle at P, and let PM be drawn perpendicular to the diameter ACB, C being the centre of the circle. At A, C, B, draw lines perpendicular to the radius meeting the tangent at P in A′, C′, B′. Then AA′, MP, CC′, BB′ are proportional. Draw A′R, PQ parallel to AB and meeting PM, BB′ in R, Q respectively. Then by the similar triangles PA′R, B′PQ, the required proportion may be deduced, observing that A′A is equal to A′P; B′P to B′B, and CC′ an arithmetic mean between AA′ and BB′.

99. The lines so drawn may be proved by Euc. VI. 3, to be proportional to the segments of the base of the triangle SEL, Theorem 123, p. 361.

100. Let any tangent to the circle at E be terminated by AD, BC tangents at the extremity of the diameter AB. Take O the centre of the circle and join OC, OD, OE: then ODC is a right-angled triangle and OE is the perpendicular from the right angle upon the hypothenuse.

101. Let BA, AC be the bounding radii, and D a point in the arc of a quadrant. Bisect BAC by AE, and draw through D, the line HDGP perpendicular to AE at G, and meeting AB, AC, produced in H, P. From H draw HM to touch the circle of which BC is a quadrantal arc; produce AH, making HL equal to HM, also on HA, take HK equal to HM. Then K, L, are the points of contact of two circles through D which touch the bounding radii, AB, AC.

Join DA. Then, since BAC is a right angle, AK is equal to the radius of the circle which touches BA, BC in K, K′; and similarly, AL is the radius of the circle which touches them in L, L′. Also, HAP being an isosceles triangle, and AD is drawn to the base, $AD^2$ is shewn to be equal to AK . KL. Euc. III. 36; II. 5, Cor.

102. The radius of the second circle is half the radius of the first, the radius of the third half that of the second circle, and so on. The radii of the second, third, &c. circles form a Geometric series.

103. Let O, O′ be the centres of the inscribed and escribed circles. Join $OD_1$, OB, O′B, $OD_2$. Then the triangles $OBD_1$, $O'BD_2$ may be shewn to be similar; whence may be shewn $BD_1 \,.\, BD_2 = R \,.\, r$. And by joining OE, OC, $O'C_1$, $O'E_2$, in a similar way may be shewn, $R \,.\, r = CE_1 \,.\, CE_2$.

104. By the point C is to be understood that in which BD cuts the quadrant ACB. Complete the semicircle BAG, BG being the diameter and E the centre. Join AC, AG; then by means of the quadrilateral ACBG inscribed in the circle, DCA may be shewn to be half a right angle: also ADB a right angle subtended by AB. Hence the locus of D is a semicircle; and the ratio of AB to BG may be shewn to be as $1 : \sqrt{2}$.

105. Through E one extremity of the chord EF, let a line be drawn parallel to

one diameter, and intersecting the other. Then the three angles of the two triangles may be shewn to be respectively equal to one another.

106. This is a repetition of Theorem 99, p. 359, under a somewhat different form of expression.

107. Let the tangents at A and C, A and B, B and C, meet in the points G, H, K, respectively.

Since the transversals FC, DA, EA intersect the triangle HKG,

$$\frac{GF}{FH}\cdot\frac{HB}{BK}\cdot\frac{KC}{CG}=1;\quad \frac{HA}{AG}\cdot\frac{GD}{DK}\cdot\frac{KB}{BH}=1;\quad \frac{HE}{EK}\cdot\frac{KC}{CG}\cdot\frac{GA}{AH}=1;$$

and observing that HA = HB, KB = KC, GC = GA;

$$\text{hence } \frac{GF}{FH}\cdot\frac{HE}{EK}\cdot\frac{KD}{DG}=1,$$

which is the condition fulfilled when a transversal intersects the three sides AB, AC, CB produced of the triangle, in the points D, E. F. (See Appendix, p. 22.)

108. It may be proved that the vertices of the two triangles which are similar in the same segment of a circle, are in the extremities of a chord parallel to the chord of the given segment.

109. This is the converse to Euc. VI. D, and may be proved indirectly.

110. Since the lines joining B, C, D, are equal, the consequence is obvious from the Proposition.

111. Perhaps the simplest mode of shewing the truth of this property is by means of transversals. The triangles CBP, BAQ are cut by the transversals ADQ, CDP, respectively.

Whence BA . PD . CQ = AP . DC . QB, and BC . QD . PA = CQ . DA . PB: from which the required proportion may be deduced.

112. Let the line AD drawn from the vertex A, meet the base of the isosceles triangle ABC in D, and let AD produced meet the circumference of the circumscribed circle in E. Then by Euc. II. 3; III. 35, and Theorem 27, p. 309.

113. This follows at once from Euc. III. 36, 37.

114. Correct the enunciation thus:—Let AD, FC, meet in P, and AE, BK in Q: then the points P, Q, B, C, E, D are in the circumference of one circle. For let FC meet AB in R. Then it has been proved, Euc. I. 47, that the angle BFR is equal to RAP. Also the angles FRB, ARP are equal; wherefore the angles FBR, APR are equal, and hence APR is a right angle. Whence, again, DPC is a right angle, and equal to DBC. It is hence in the same semicircle on DC; that is, in the circle BCED. In the same manner Q may be shewn to be in the same circle BCDE.

115. Let ABC be a triangle, and let the line AD bisecting the vertical angle A be divided in E, so that BC : BA + AC :: AE : ED. By Euc. VI. 3 may be deduced BC : BA + AC :: AC : AD. Whence may be proved that CE bisects the angle ACD, and Euc. IV. 4, that E is the centre of the inscribed circle.

116. For "bisected," read "divided into parts, one of which is double the other, the smaller segment being estimated from the centre of the circle."

Let ABC be the triangle; Q the centre of the circumscribing circle; P the intersection of the perpendiculars BG, CH: D, E the middle points of BA, CA; divide PQ in R, so that PR = 2 . QR; and join BR, RE: also draw DQ, QE, ED. Then the triangles BPC, EQD may be shewn to be equiangular, and hence BP = 2 . QE.

Again, PQ meeting the parallels QE, BP, the angles RQE, RPB are equal; and by hypothesis RP = 2 . RQ: whence the sides about the equal angles are proportional, that is, EQ : QR :: BP : PR, and the angles QRE, PRB are equal. The points B, R, E are therefore in one line. The same triangles give PR : RQ :: BR : RE, and hence BR = 2 . RE; or the point R is distant from B, two-thirds of the line BE drawn to the middle of the opposite side AC.

117. If the extremities of the diameters of the two circles be joined by two straight lines, these lines may be proved to intersect at the point of contact of the two circles; and the two right-angled triangles thus formed may be shewn to be similar by Euc. III. 34.

118. From the centres of the two circles let straight lines be drawn to the extre-

mities of the sides which are opposite to the right angles in each triangle, and to the points where the circles touch these sides. Then by similar triangles.

119. Let DB, DE, DCA be the three straight lines, fig. Euc. III. 37; let the points of contact B, E be joined by the straight line BC cutting DA in G. Then BDE is an isosceles triangle, and DG is a line from the vertex to a point G in the base. And two values of the square of BD may be found, one from Theo. 27, p. 309; Euc. III. 35; II. 2; and another from Euc. III. 36; II. 1. From these may be deduced, that the rectangle DC, GA is equal to the rectangle AD, CG. Whence the, &c.

120. Let the arc AE be double the arc AB of the circle whose centre is C. Let CD, CF, be the perpendiculars on the chords of the arcs AB, AE. Produce CF to meet the circumference in B and G, join GA and draw CH perpendicular to GA. The proportion is deduced from the similar triangles CBD, GFA.

121. Draw FG to bisect the angle DFE, and draw DK, EH perpendicular on FG; and let FK meet AB in G.

$$\text{Then } 2.GB : BF :: 2.HE : FE, \text{ and } 2.AG : AF :: 2.DK : FD;$$

by similar triangles: and by compounding these proportions, observing that AF = FB, AG = GB, and 4.HE.DK = 4.DC.CE, there results

$$AB^2 : AF^2 :: 4.EC.CD : FD.FE; \text{ similarly } BC^2 : CE^2 :: 4.FA.AD : FE.ED;$$

$$\text{whence } AB^2 : BC^2 :: DE.FA : EC.DF.$$

122. This is manifest from Euc. III. 36, 37.

123. Join OE. Then OE is equal to OA, and Euc. VI. 6, the triangles OES, OLE are equiangular. Whence it may be shewn that the angle SEL is bisected by EA.

124. Let BD touch the inner semicircle in E, and let O be its centre. Join OE. The triangles DAB, EOB are equiangular.

125. Let ABC be the triangle, and F a point in its base BC;

let the circles AFB, AFC be described, and their diameters AD, AE, be drawn;

then DA : AE :: BA : AC.

For join DB, DF, EF, EC, the triangles DAB, EAC may be proved to be similar.

126. In the enunciation, for "two circles" read "two equal circles, whose centres lie on opposite sides of the line ABCD."

The proof offers no difficulty. In every other case the theorem does not hold good.

127. Let the figure be constructed, and the similarity of the two triangles will be at once obvious from Euc. III. 32; Euc. I. 29.

128. Let the figure be drawn, and BC, CD, BD be joined. Then ABCD is a quadrilateral figure inscribed in a circle, and BD, AC are the diagonals. By Euc. VI. D, 17, the first proportion is deduced; and the other in a similar way.

129. Let the figure be drawn, and join HI. Then EF, HI are parallel to KN, a side of the triangle BKN. Euc. III. 37; VI. 2.

130. Let O be the centre of the inscribed circle DEF, and P that of the escribed circle HIK; these are in the line bisecting the angle C. Join MB, LA cutting COP in N and R; draw the several radii to the points of contact; and join OA, OB, PA, PB. Then prove that FK is equal to the difference of the sides AC, CB; and therefore to AM. Next, the lines BM, AL are perpendicular to CP, which bisects the common vertical angle, and CNB, CRL are right angles, as are also the angles made by OF, AB. Describe semicircles about ONFB and OFRA, and join NF, RF. Then the angle AFR = AOR = BOF = BNF; and the alternate angles FAR, FBN are equal. The triangles AFR, BNF are therefore equiangular, and AR : AF :: FB : BN; also 4.AF.FB = 4.AR.BN = AL.BN.

131. By Euc. IV. 4, twice the area of the triangle is equal to the rectangle contained by the sum of the sides and the radius of the inscribed circle. By Theorem 137, p. 362, the area is expressed in terms of the sides and the radius of the circumscribed circle. Whence the property required may be deduced, observing that one of the sides of the triangle is half the sum of the other two sides.

132. Since the line $mnp$ is a transversal to the triangle ABC; $An.Cp.Bm = nC.pB.mA$; and by Euc. III. 36, the values of $t_p^2$, $t_m^2$, $t_n^2$ may be expressed in terms of $An$, $nC$, &c.: whence the property may be deduced.

133. Let A be the centre of the circle whose circumference passes though B the centre of the other circle, and CD the line which joins the intersections of the two

circles; draw BF cutting the first circle in F and the second in E, and draw EG perpendicular to CD: then FE : EG is a given ratio.

For, join BC, CF, FD, CE, and draw EP perpendicular to CF. The angle FCE may be shewn to be equal to the angle DCE; or EC bisects the angle FCD, and hence EG is equal to EP. But the angle FEG being given, and EPF being a right angle, the ratio FE to EP is given; that is, the ratio FE to EG is given.

134. See Theorems 153, 154, infra.

135. Let the sides of the triangle ABC be divided in the ratio of $n$ to 1 in the points D, E, F. Join DE, EF, FD. Then the ratios of each of the triangles ADF, BDE, CEF to the triangle ABC may be found by Theorem 85, p. 358, in terms of $n$, whence also the ratio of the triangle DEF to the triangle ABC in terms of $n$.

136. Let the chords AB, CD intersect each other in E, so that AE is to EB as CE to ED. Then it may be shewn that the lines joining DB, AC are parallel, and that the line bisecting the angle at E bisects these parallels.

137. By Euc. VI. E. $BA.AC = EA.AD$. Multiply these equals by BC, and interpret the result.

138. For let the circle be described about the triangle EAC, then by the converse to Euc. III. 32, the truth of the proposition is manifest.

139. The triangles ABC, ADB may be shewn to be equiangular.

140. Let the figure be constructed as in Theorem 3, p. 313, the triangle EAD being right-angled at A, and let the circle inscribed in the triangle ADE touch AD, AE, DE in the points K, L, M respectively. Then AK is equal to AL, each being equal to the radius of the inscribed circle. Also AB is equal to GC, and AB is half the perimeter of the triangle AED.

Also if GA be joined, the triangle ADE is obviously equal to the difference between the figure AGDE and the triangle GDE, and this difference may be proved equal to the rectangle contained by the radii of the two circles.

141. Let the figure be constructed, then from the isosceles triangles, ED is shewn to be equal to EA, and EG to EB. Then Euc. VI. 13.

142. Let FG join the intersections of the circles, and cut AE in C.

Then, $AC.CD = FC.CG = BC.CE$, or, $AC : CE :: BC : CD$; whence,

$$AC+CE : BC+CD :: AC : CB, \text{ and } AC+CE : BC+CD :: CE : CD;$$

compounding these proportions and putting for $AC+CE$ and $BC+CD$, their equals, we have $AE^2 : BD^2 :: AC.CE : BC.CD$.

143. Let ABC be any triangle, and let D, E be the centres of the circumscribed and inscribed circles respectively. Join AD, and through D draw the diameter FDG and join AE; AE produced meets the diameter in F. Draw EH perpendicular to AC and join DE, EC, FC, CG. Then FC is equal to FE, and by Theorem 27, p. 309, $DE^2 = DA^2 - AE.EF = DA^2 - AE.FC$; also the triangles AEH, GFC being similar, $AE.FC = GF.EH$. Whence the truth of the theorem may be shewn.

144. This property follows directly from Euc. VI. C.

145. Join DC, then in the triangles ADB, ADC, the angle ACD is equal to the angle ADB, both standing upon equal arcs of the same circle, also the angle DAB is common to the two triangles. Hence the triangles are equiangular, and by Euc. VI. 4, the property is manifest.

146. Let ACB be the common diameter of the two circles which touch each other in the point A, and through C the centre of the smaller circle, let PP′ be drawn perpendicular to AB, and meeting the inner circle in Q, Q′: also let the tangents from P, P′ touch the inner circle in T, T′. Join CT, CT′. Then PT, P′T′, may be proved each equal to CT, CT′.

147. Let the circles cut each other in A, B, join AB, and on AB as a diameter describe a circle cutting the two given circles, and from A draw a straight line ACDE meeting the circumference in C, D, E. From B the other extremity of the diameter, draw BF, BG perpendicular to AB and meeting the circumferences of the two given circles in F, G. Then CD is to DE as BG to BF. The triangles CDB, ABG are similar, as also the triangles BED, ACF.

148. Half the difference between the sums of the opposite sides is equal to the distance between the points where the two circles touch one of the sides of the figure. This distance may be proved to be a mean proportional between the diameters of the circles.

149. Let the diagram be constructed according to the enunciation; and let PP′ be the common tangent; draw PS parallel to OO′, (O, O′ being the centres) meeting OP in S; produce PO, P′O′ to meet DE, D′E′ in Q, Q′. Then the triangles ABC, ABC′ are equiangular, whence BC : BA :: BA : BC′. Again join DD′ and bisect DD′ in M, and make $Mm$ = half the difference of the sides AD, AD′. Then DE . D′E′, and $PP'^2$, may be each shewn to be equal to $4r^2-(R-r)^2$, R, $r$ being the radii of the two circles.

150. Let AB be a line given in position, and P the given point. At P make the angle APD equal to the given angle, taking the point D, so that AP is to PD in the given ratio.

Draw a line through the point D making with PD an angle equal to the angle PAB. Then this line is the locus of the extremity D of the line PD. This may be proved by taking another point A′ in the given line, and making the angle A′PD′ equal to the angle APD. Then the triangles A′PA, D′PD may be shewn to be similar.

151. By constructing the figure and joining AC and AD, by Euc. III. 27, it may be proved that the line BD falls upon BC.

152. Join AG, AF, AD. Then, since the circles are equal, the segments AFB, AGB contain supplementary angles. Whence the angles AFG, AGF are equal, and AF is equal to AG. Again, AEB is a right angle; and hence AE being perpendicular to the base GF of the isosceles triangle, bisects FG. See Prob. 23, p. 316.

In the enunciation, "a third circle drawn with centre A," &c. appears to be superfluous.

153 and 154 may be taken together, with a few other properties, some of which, however, have been noticed in other places.

(1) Construct according to the enunciation, and complete the diameter AB through A; since

OS : OA :: OA : OL, we have OA − OS : OS + OA :: OL − OA : OL + OA, or since OA = OB, this becomes AS : SB :: AL : LB, which expresses that LB is harmonically divided in S and A.

(2) Join PO, QO; since PO = OA, therefore OS : OP :: OP : OL; and the triangles POS, LOP having the angle at O common and the sides about that angle proportionals, they are similar. Whence OS : OP or OA :: SP : PL; or the ratio of SP : PL is constant. In like manner the ratio SQ : QL is also constant, and the same as SP : PL. This is Theorem 154, (1).

(3) Join AP, BP; then since OS : OA :: OA : OL, hence

OS : OA :: OA − OS : OL − OA :: SA : AL,

OS : OA :: OA + OS : OL + OA :: BS : BL.

Whence SA : AL :: SP : PL, and SB : BL :: SP : PL; and the lines AP, BP bisect the interior and exterior angles SPL, SPQ of the triangle SPL at P.

Similarly, QA, QB bisect the exterior and interior angles of the triangle SQL at Q.

(4) Let the perpendicular at S produced meet the circumference in C, D: and join OC, CL: since OS . OL = $OA^2$ = $OC^2$, and OSC is a right angle, it follows that OCL is also a right angle, and that LC is a tangent to the circle at C.

In the same way it may be shewn that LD is a tangent at D. Whence the tangents at C and D meet the diameter AB produced in the same point.

(5) By the right-angled triangles OCL, CLS; OL . LS = $CL^2$ = PL . LQ; or the four points P, Q, O, S are in the circumference of a circle. Whence the exterior angle PSA of the quadrilateral is equal to the opposite angle OQP. But by the similar triangles OQS, OLQ, the angle OSQ = OQL. Whence QSP = ASP; and QSC = PSC.

(6) Produce QS to meet the circle in F: then ASF = QSB = QSA; and hence SP = SF; wherefore PS . SQ = SF . SQ = $SC^2$, a constant magnitude. This is 154, (2).

(7) Since SE bisects the interior angle PSQ, and SL the exterior angle PSF, of the triangle PSQ, PE : EQ :: PS : SQ :: PL : LQ; or LQ is harmonically divided in P, E. This is Theorem 153.

(8) Produce AP, BQ to meet in G, and let AQ, BP meet in H; then G, H will be in the line CD. For in the triangle PSQ, the three lines QA, BP, SC have been shewn to bisect the angles; wherefore these lines meet in a point.

Also, SC bisects PSQ, and AP, BQ bisect the exterior angles at P and Q; therefore they also meet in a point. Whence G, H are in the line CD.

(9) The lines BG, AG, are bisected in K and I by the circle which passes through the points P, S, O, Q.

(10) Let the circle PSOQ cut SG in M; and draw MO, MQ, MP.

Then since SM bisects the angle QSP, it bisects the circumference QMP on which QSP stands; and hence MQ = MP. Also, since QO = OP, it follows that MO is perpendicular to QK, and is a diameter of the circle PSOQ. Whence OQM, OPM are right angles. But OQ, OP are radii of the circle BQPA, and hence QM, PM are tangents at Q and P; and they meet at M in SC produced. Wherefore tangents at P and Q always meet in the line SC produced. This is 154, (3).

(11) If any chord QSF be drawn through the pole S, and QL, FL be drawn; then the angle SLF = SLQ.

For, join SP: then QS . SP = AS . SB = QS . SF: whence SF = SP, and the triangles PLS, FLS are equal in all respects, and hence the angle SLF = SLQ.

(12) Conversely. If QF be drawn through S, and lines be drawn from Q, F to make equal angles with LV drawn through L; the line which bisects the angle FLQ is a diameter passing through S.

Note. The line SG is called the *polar*, and L the *pole*; as are also the line LV and the point S, so called respectively. Taken together, either point and its respective line are called *reciprocal polars*;—as for instance SG and L.

The characteristic property of the *pole* and *polar* to which it is most convenient to refer, is, that if the diameter of a circle AB be produced and be harmonically divided in S and L; then a perpendicular to AB through S is the polar of L, and a perpendicular to AB through L is the polar of S.

155. Let AB be that diameter of the given circle which when produced is perpendicular to the given line CD, and let it meet that line in C; and let P be the given point: it is required to find D in CD, so that DB may be equal to the tangent DF.

Make BC : CQ :: CQ : CA, and join PQ; bisect PQ in E, and draw ED perpendicular to PQ meeting CD in D: then D is the point required. Let O be the centre of the circle, draw the tangent DF; and join OF, OD, QD, PD. Then QD may be shewn to be equal to DF and to DP.

When P coincides with Q, determined as in the construction, *any* point D in CD fulfils the conditions of the problem: that is, there are innumerable solutions.

156. Apply the method of transversals. See Appendix, p. 15, &c.

157. By assuming Euc. VI. 1, it may be shewn that

$$\frac{\text{figure ABPCA}}{\text{triangle AMN}} = \frac{MP}{PN} + \frac{PN}{MP}.$$

Also by theorem 85, p. 358, that $\dfrac{\text{triangle ABC}}{\text{triangle AMN}} = 2 + \dfrac{MP}{PN} + \dfrac{PN}{MP}$. Whence it follows that the triangle BPC is twice the triangle AMN. What is the fixed point?

158. It may be shewn by the theory of transversals, that when the three lines drawn from the angles to the opposite sides of a triangle, pass through any point P, $Ab . Bc . Ca = aB . bC . cA$. By means of Euc. III. 36; it may be shewn, that

$$\frac{Ab'}{b'C} \cdot \frac{Ca'}{a'B} \cdot \frac{Bc'}{c'A} = \frac{Ab}{bC} \cdot \frac{Ca}{aB} \cdot \frac{Bc}{cA}.$$

The second part is to be shewn indirectly, by supposing one of the lines from the angle C, to intersect the opposite side in some other point $c'$ different from $c$, and then shewing that $c$, $c'$ coincide.

159. First. Join FA, AH: and prove that F, A, H, are in one line. Secondly. Join BG, HC; and prove that BG, HC are parallel, and GH, BC equal. Thirdly. Produce GH, BC to meet in M; and shew that the triangles BMG and CMH are isosceles. Lastly. Prove that FK bisects their bases, GB, HC at right-angles; and hence coincides with the perpendicular from M to the base in both triangles.

160. See Appendix, p. 18.

161. See Appendix, p. 16.

162. Let DG, EH, FK, be drawn perpendicular to the middle of the sides BC,

CA, AB of the triangle ABC, and each equal to half the side from the middle of which it is drawn; join GH, HK, KG, AK, KB, BG, GC, CH, and HA; produce GC, and draw HQ perpendicular to it, and AP perpendicular to BC. Then the two right-angled triangles APC, CQH, are similar; whence $AP^2 = 2 . CQ^2$; also $GC^2 = 2 . CD^2$: and Euc. II. 12, $GH^2 = GC^2 + CH^2 + 2 . GC . CQ$, which may be shewn to be equal to $\frac{1}{2} . CB^2 + \frac{1}{2} . CA^2 + 2 .$(triangle ABC). Similarly for $HK^2$, $KG^2$. Whence $GH^2 + KG^2 + HK^2$ is found.

163. Let E, F, G be the centres of the circles inscribed in the triangles ABC, ADB, ACD. Draw EH, FK, GL perpendiculars on BC, BA, AC respectively, and join CE, EB; BF, FA; CG, GA. Then the relation between R, $r$, $r'$, or EH, FK, GL may be found from the similar triangles, and the property of right-angled triangles.

164. Let ABC be any triangle, and from A, B let the perpendiculars AD, BE on the opposite sides intersect in P: and let AF, BG drawn to F, G the bisections of the opposite sides, intersect in Q. Also let FR, GR be drawn perpendicular to BC, AC and meet in R: then R is the centre of the circumscribed circle. Join PQ, QR, then PQ is in the same straight line as QR.

Join FG, and by the equiangular triangles GRF, APB, AP is proved double of FR. And AQ is double of QF, and the alternate angles PAQ, QFR are equal. Hence the triangles APQ, RFQ are equiangular, therefore, &c.

165. Let ABC be a triangle, D the centre of the inscribed circle, draw DE perpendicular to the base BC, E is the point where the inscribed circle touches the base. Join CE and bisect it in F, bisect also BC in G. Then the points G, D, F may be proved to be in a straight line.

Draw GH perpendicular to BC, and DK perpendicular to GH. Join CK and produce it to meet BC in L, join also GD, DF. Then LG is equal to GE, and by the similar triangles CE may be proved to be bisected in F. Hence G, D, F must be in the same straight line.

166. This question is rather Algebraical than Geometrical. If the expression for the area of a triangle in terms of the sides may be assumed, the equation which connects the radii of the three circles may be deduced from the expression for the area of a triangle in terms of the sides and the radius of the inscribed circle, and theorem 137, p. 362.

167. From Euc. IV. 6, and Theo. 1, p. 332, the arithmetical ratio of the sides may be found: also the ratio of their areas.

168. Let a perpendicular be drawn from the centre C on $ab$, AB two sides (supposed parallel) of the inscribed and circumscribed polygons meeting them in $d$, D. Join D$a$: then D$a$ is a side of a regular inscribed polygon of double the number of sides. The areas of the three polygons are the same multiples of the triangles $ad$ C, $a$DC, ADC, which may easily be shewn to have the proportion stated.

169. The first property may be perhaps more clearly seen by first shewing it to be true in a figure of three sides, then in one of four sides, &c.; and lastly in any polygon whatever. As all the figures described upon the parts of one side are similar to the given figure, any two sides of the given figure are proportional to the homologous sides of the smaller figures, and hence the property may be shewn respecting the sides. Also the second property may be proved by Euc. VI. 20.

170. Each pentagon can be divided into the same number of triangles, and by Euc. VI. 19.

171. The triangles which form the two regular figures are the same in number, and are similar. By the similarity of triangles the proportion is obvious.

172. The area of the inscribed equilateral triangle may be proved to be equal to half the inscribed hexagon, and the circumscribed triangle equal to four times the inscribed triangle.

173. See Theorem 168, supra.

## HINTS, &c. TO THE PROBLEMS. BOOK XI.

1. See Géométrie par A. M. Legendre, (dixième édition), p. 156.

2. Describe a circle passing through the three given points, and from the centre draw a line perpendicular to its plane. Then every point in this perpendicular fulfils the conditions required.

3. This is an indeterminate Problem. If however, the circle be in that plane which passes through the given point, and be perpendicular to the two given planes, the problem is reduced to that of describing a circle which shall pass through a given point, and touch two given straight lines.

4. Let ACB, ADB, be the triangles, CD being perpendicular to the plane of ACB. Then the angles DAC, DBC, and the line DC being given, the lines DA, DB, can be constructed. Draw $ab$ through C perpendicular to CD, and make the angles C'D$a$, C'D$b$, the complements of the given angles; then C$a$, C$b$ will be of the same length as CA, CB. Whence the angle which they are to form being given, the triangle may be constructed, and its base is the line required.

5. About the given line let a plane be made to revolve, till it passes through the given point. The perpendicular drawn in this plane from the given point upon the given line is the distance required.

6. Let A, B, be the given points, and GH the given straight line; draw AC, BD perpendicular on GH, and in the plane AGH produced, draw DB′ perpendicular to GH, and equal to DB: join AB′, meeting GH in E, and draw EB. Then AE + EB is the minimum. For the triangles EDB, EB′D are equal, being right-angled at D, and having one side common, and the others equal. Whence the angle BEH is equal to GEA, each being equal to B′EH. The conclusion follows from the demonstration of Theorem 17, p. 369.

7. Through any point in the first line draw a line parallel to the second; the plane through these is parallel to the second line. Through the second line draw a plane perpendicular to the forenamed plane cutting the first line in a point.

Through this point draw a perpendicular in the second plane to the first plane, and it will be perpendicular to both lines.

8. Through any point draw perpendiculars to both planes; the plane passing through these two lines will fulfil the conditions required.

9. Let ABCD be the given triangular pyramid, of which the vertex is A, and base, the triangle BCD. Bisect BC in E, and join AE, BE: then AEB is an isosceles triangle, having its base AB equal to the side of the equilateral triangle, and its sides equal to the perpendicular from the vertex on the base. From A draw AF perpendicular to BE: AF is the perpendicular required.

Or thus. Let ABC be the equilateral triangle which forms the base of the pyramid, and D its centre. On AC describe a semicircle, and make AE equal to AD. Join AE, then AE is the perpendicular altitude.

From D draw the perpendicular DF equal to CE, and join FC, FA, FB. Then the triangle AFD is equal in all respects to the triangle ACE, and hence AF is equal to AC. In like manner the sides FC, FB are each proved equal to a side of the equilateral triangle ACB.

10. Through each line draw a plane parallel to the other; these planes will be parallel, and obviously form two of the faces of the parallelopiped. Through each line and one extremity of the other, draw a plane; and a second plane parallel to it through the remaining extremity. This will complete the figure; but there will be four varieties of cases according as the extremities are situated.

11. Every possible combination of the lines taken three at a time will form the pyramids, since the respective faces may be so formed. These according to the ordinary method are fifteen in number.

12. Bisect the base by a line drawn in the given direction, whether parallel to a given line, or tending to a given point. The plane drawn through the bisecting line and the vertex of the pyramid, gives the solution of the problem.

13. The section will be in all cases a parallelogram, but not necessarily rectangular. Any plane drawn perpendicular to a face through a line equal to the edge of the cube, will fulfil the condition.

## HINTS, &c. TO THE THEOREMS. BOOK XI.

3. Let AD, BE be two parallel straight lines, and let two planes ADFC, BEFC pass through AD, BE, and let CF be their common intersection, fig. Euc. XI. 10. Then CF may be proved parallel to BE and AD.

4. See the figure Euc. XI. 10. If the lines be unequal, the figure may be shewn not to agree with the definition of a prism. Euc. XI. def. 13.

5. This theorem is analogous to Euc. XI. 8. Let two parallel lines AC, BD meet a plane in the points A, B. Take AC equal to BD and draw CE, DF, perpendiculars on the plane, and join AE, BF. Then the angles CAE, DBF, are the inclinations of AC, BD to the plane, Euc. XI. def. 5, and these angles may be proved to be equal.

6. Let lines be drawn in each plane through the points where the lines cut the planes, then by Euc. I. 29.

7. AB, BC, CD may be shewn to be three consecutive edges of a rectangular parallelopiped, of which AD is the diagonal.

8. If the intersecting plane be perpendicular to the three straight lines, by joining the points of their intersection with the plane, the figure formed will be an equilateral triangle. If the plane be not perpendicular, the triangle will be isosceles.

9. Let AB, CD be parallel straight lines, and let perpendiculars be drawn from the extremities of AB, CD on any plane, and meet it in the points A′, B′, C′, D′. Draw A′B′, C′D′, these are the projections of AB, CD on the plane, and may be proved to be parallel.

10. Let AE meet the straight lines BE, DE in the plane BED, fig. Euc. XI. 6, and let the angle AEB measure the inclination of AE to the plane BDE; then the angle AEB is less than the angle AED. Draw AB perpendicular to the plane, make ED equal to EB and join BD, AD. Euc. I. 18, 19.

11. Let the three parallel straight lines AD, BE, CF be cut by the parallel planes ABC, DEF, and A, B, C, the points of intersection of the lines, be joined, as also D, E, F: then the figure ABC may be proved to be equal and similar to the figure DEF.

12. Let AB be at right angles to the plane BCED, and let the perpendiculars from AB intersect the plane GHKL in the line MN, and let HNK be the common intersection of the planes CBDE, GHKL. Join AM, BN, and prove MN to be a straight line perpendicular to HK.

13. This may be readily proved by Euc. XI. 17.

14. Let AB, A′B′ be any portions of the two straight lines. At B′ draw B′C′ parallel to AB, and B′C perpendicular to the plane passing through A′B′C′. Let the plane passing through A′B′C intersect the line AB in the point A. In the plane A′B′C from A draw AA′ perpendicular to A′B′, and AC perpendicular to AA′. Then the plane CAB passing through the line AB may be shewn to be parallel to the plane A′B′C′ passing through the line A′B′, and that no other parallel planes can be drawn through AB, A′B′. Also AA′ is the perpendicular distance between the two planes, and that AA′ is less than any other line which can be drawn between the two planes.

15. The figure formed by lines drawn from a point above the plane of a circle to every point in its circumference is a cone. If the point be in the perpendicular to the plane drawn from the centre of the circle, the cone is a right cone, and all lines from the point to the circumference are equal; if the point be not in the perpendicular, the cone is an oblique cone and has only the two lines equal, as may readily be shewn.

16. Let BC be the common intersection of the two planes ABCD, EFGH which are inclined to each other at any angle. From K at any point in the plane ABCD, let KL be drawn perpendicular to the plane EFGH, and KM perpendicular to BC, the line of intersection of the two planes. Join LM, and prove that the plane which passes through KL, KM is perpendicular to the line BC.

17. Let GH be the edge of the wall, A, B the two points, and let the line joining A, B meet the edge of the wall GH in E. If the points AE, BE make equal

angles with GH, then AE, EB may be proved to be less than any other two lines drawn from A, B, to meet GH in any other point E'.

18. Let AB, AC drawn from the point A, and A'B', A'C' drawn from the point A', in two parallel planes, make equal angles with a plane EF passing through AA', and perpendicular to the planes BAC, B'A'C'. Let AB in the plane ABC be parallel to A'B' in the plane A'B'C'; then AC may be proved to be parallel to A'C'.

19. Let HM be the common section of the two planes MN, MQ; and let AB be drawn from a point A in HM perpendicular to the plane MN: then, if planes be drawn through AB to cut the planes MN, MQ in lines which make the angles CAD, EAF with each other, and that the plane BACD is perpendicular both to MN and MQ, the angle CAD will be greater than EAF. Shew that the angle BAD is less than the angle BAF, and it follows that CAD is greater than EAF.

20. Let the depth be taken as the fixed unit, and let the breadth be double the depth, and the length double the breadth. Let the parallelopiped and the cube be constructed. Then the cube may be shewn to consist of the same number of cubic units as the parallelopiped. See note on def. A, p. 253.

Similarly, if the breadth be treble, or any other multiple of the depth, and the length treble, or the same multiple of the breadth, the equality of the two figures may be shewn to exist.

21. This theorem is analogous to the corresponding theorem respecting a rectangular parallelogram.

The axis of a parallelopiped must not be confounded with its diagonal.

22. This theorem is analogous to Euc. II. 4.

23. There is some inaccuracy in the enunciation of this theorem.

24. Let O be one of the solid angles of a cube whose three adjacent edges are OA, OB, OC; OBEC being the base of the cube. On OA, OB, OC, let any three points A', B', C' be taken, and join A'B', B'C', C'A'. Then the square of the area of the base A'B'C' of the solid OA'B'C' is equal to the sum of the squares of the areas of the faces, OA'B', OA'C', OC'B'.

Join OE intersecting B'C' in E', and join A'E'. Then A'E' may be shewn to be perpendicular to the base C'B' of the triangle A'B'C', and by Euc. I. 47, and note page 68, the truth of the property is shewn.

25. Let the figure be described, then in a similar manner to Theorem 2, page 367, by employing Euc. II. 12, 13, instead of Euc. I. 47, the truth of the theorem may be proved.

26. This is to shew that the square of the diagonal of a rectangular parallelopiped is equal to the sum of the squares of its three edges.

27. Let a rectangular parallelogram ABCD be formed by four squares, each equal to a face of the given cube, and let EF, GH, KL, be the lines of division of the four squares. Let BD the diagonal of ABCD, cut EF in M; the square of BM to the square of AB is as 17 to 16. Let BG the diagonal of ABHG cut EF in N; the square of BN is to the square of AB, as 20 is to 16: hence there is some square between that of BM and BN which bears to the square of AB, the ratio of 18 to 16, or of 9 to 8.

28. If any point A in a sheet of paper be taken as the vertex of any pyramid (suppose a triangular pyramid), the three plane angles which can be formed at A, are equal to four right angles, and therefore greater than the sum of the three plane angles with which it is possible to form a solid angle.

29. Let BCD be the base of the pyramid. Take C'D' equal to CD in the same line, and join AC', BC', AD', BD'. Then the triangular base BC'D' is equal to BCD, Euc. I. 38. And since A is a fixed point, the altitude of the pyramids ABCD, ABC'D' is the same, and pyramids of the same altitude on equal bases are equal.

30. See Euc. VI. def. 1. From the vertex A draw a line to any point B in the base of the pyramid, and meeting the given section in B'. From the angular points of the base draw lines to the point B; also from the angular points of the given section to the point B'. Then any triangle in the section, may be shewn to be similar to the corresponding triangle in the base. Euc. VI. 20.

## HINTS, &c. TO THE PROBLEMS. BOOK XII.

1. There is no method by which a square can be described, by plane geometry, exactly equal to the area of a circle. The tract of Archimedes on the mensuration of the circle, consists of the three following propositions. 1. Every circle is equal to the right-angled triangle whose base and perpendicular are equal to the radius and circumference of the circle. 2. The area of a circle is to the square described on its diameter as 11 to 14 nearly. 3. The circumference of a circle is equal to three times the diameter and a part of the diameter which is less than $\frac{10}{70}$ of the diameter, but greater than $\frac{10}{71}$ of the diameter.

2. The radius of a circle whose area is double that of another, is equal to the side of a square whose area is double that of the square of the radius of the given circle.

3. A similar remark applies here as to the preceding problem.

4. First, to bisect a circle by a concentric circle. Let C be its centre, AC any radius. On AC describe a semicircle, bisect AC in B, draw BD perpendicular to AC, and meeting the semicircle in D; join CD, and with centre C, and radius CD, describe a circle; its circumference shall bisect the given circle. Join AD. Then by Euc. vi. 20, Cor. 2, the square on AC is to the square on CD as AC is to CB: and Euc. xii. 2. In the same way, if the radius AC be trisected, and perpendiculars be drawn from the points of trisection to meet the semicircle in D, E, the two circles described from C with radii CD, CE shall trisect the circle. And generally, a circle may be divided into any number of equal parts.

Note. By a similar process a circle may be divided into any number of parts which shall have to each other any given ratios.

5. To divide the circle into two equal parts. Let any diameter ACB be drawn, and two semicircles be described, one on each side of the two radii AC, CB: these semicircles divide the circle into two equal parts which have their perimeters equal. In a similar way a circle may be divided into three parts, by dividing the diameter into three equal parts, AB, BC, CD; and describing semicircles upon AB, AC on one side of the diameter, and then semicircles upon DC, DB on the other side of the diameter.

6. By Euc. xii. 2. The squares of the radii of the two circles may be shewn to be in the ratio of 3 to 1.

7. The area of the circle of which the quadrant is given, is to the area of the circle which touches the three circles, as 36 is to 1. And the quadrant is one-fourth of the area of the circle. Hence the area of the quadrant is to the area of the circle as 9 to 1.

8. The meaning of the enunciation of this problem is not very clear.

9. By reference to Theorem 2, p. 371, and Euc. xii. 2, the parts of the diameter may be proved to bear to each other the ratio of 1 to 2.

10. If planes be drawn through the bisections of three of the edges of the tetrahedron at right angles to the edges, the point of their mutual intersection, is the centre of the sphere which circumscribes the tetrahedron.

11. Take a point A on the spherical surface of the fragment as a centre, and with any radius AB describe a circle upon it. Take two other points C, D in the circumference of this circle, and describe a plane triangle A′B′C′ having its sides equal to the distances AB, BC, CA, respectively. Describe a circle about the triangle A′B′C′, and draw the diameter A′D′; with centres A′, D′ and radius equal to AB, describe circles intersecting each other in E′, and through the points A′, D′, E′ describe a circle; the diameter of this circle will be equal to that of the sphere of which the fragment is given.

12. By Euc. i. 47, expressions for the squares of the sides of the triangle may be found, from which it will appear that the three sides of the triangle are mean proportionals between every two of the three diameters of the spheres.

13. (1) The regular tetrahedron. Each of the angles of an equilateral triangle is one third of two right angles; a solid angle may therefore be formed by three angles

of three equal and equilateral triangles, and the figure formed by the three bases of the triangles is manifestly an equilateral triangle equal in magnitude to each of the three given equilateral triangles. The angles of inclination of every two of the four faces are also equal.

(2) The regular Octahedron. Through any point O draw three straight lines perpendicular to each other, take OA, O*a*, OB, O*b*, OC, O*c* equal to one another, and join the extremities of these lines. The faces ABC, A*b*C, &c. are equilateral triangles equal to one another and eight in number: also the inclinations of every two contiguous faces are equal.

(3) The regular Icosahedron. A solid angle may be formed with five angles, each equal to the angle of an equilateral triangle. At the point A of any equilateral triangle ABC, let a solid angle be formed with it and four other equal and equilateral triangles ABD, ADE, AEF, AFC, each equal to the triangle ABC. Next at the point B, let another solid angle be formed with the triangle ABC and four others BCH, BHK, BKD, BDA, each equal to it. The solid angle at B is equal to the solid angle at A, and the inclinations of every two contiguous faces are equal; also the two solid angles have two faces ABC, ABD common. Next let a third solid angle be formed at C, by placing the two triangles CFG, CGH contiguous to the three CAB, CFA, CHB. The solid angle at C is equal to that at A or B, and the inclinations of the contiguous faces make equal angles. Thus two equal and equilateral triangles are placed contiguous one to another, forming three solid angles at A, B, C, and having every two contiguous faces equally inclined: also the solid angles formed at D, E, F, G, H, K, have alternately *three* and *two* angles of the equilateral triangles. In the same manner let another figure equal to this be formed with ten equal and equilateral triangles, each equal to the triangle ABC.

If these two figures be connected together, so that the points at which there are *two* angles of one figure, may coincide with the points which contain *three* angles of the other, there will be formed at the points D, E, F, G, H, K, six equal solid angles, each contained by five angles of the equilateral triangles, and every two contiguous faces will have the same inclination.

Hence a figure of twenty faces is formed each equal to the equilateral triangle ABC, and having the inclinations of every two contiguous faces equal.

(4) The regular Hexahedron. Since three right-angles may form a solid angle, it is therefore obvious that the solid angle formed by three equal squares, has every two of the faces equally inclined to one another; and with three other squares, each equal to the former, a figure is formed, bounded by six equal squares, and having every two contiguous faces at right-angles to one another.

(5) The regular Dodecahedron. Since three angles each equal to the angle of a regular pentagon may form a solid angle: let ABCDE be a regular pentagon, and with two others each equal to this, let a solid angle at A be formed; the inclinations of every two contiguous faces will be equal. At the points B, C, D, E successively, let solid angles be formed by pentagons equal to ABCDE. The solid angles at B, C, D, E are each equal to the solid angle at A, and the inclination of every two contiguous faces is the same. Thus is formed a figure with six equal and regular pentagons, having the inclination of every two contiguous faces equal, and the angles at the linear boundary of the figure alternately consisting of an angle of a pentagon and of two angles of two pentagons equally inclined to each other.

Next, let another figure equal to this be constructed with six pentagons, each equal to the pentagon ABCDE.

If these two figures be so placed that the angular points of the plane angles in the linear boundary of one, may coincide with the points at which there are two angles in the other figure; at each of these points will be formed ten solid angles, each equal to the angle at A, and having the inclination of every two contiguous faces equal to one another. Hence a regular figure is formed having twelve equal faces, and the inclinations of every two contiguous faces equal to one another.

14. This is repeated by mistake. It is the same as Problem 11, page 368.

15. Let A be the vertex, and BCD the triangle forming the base of the tetrahedron. Bisect each of the dihedral angles at the base by three planes which mutually intersect each other in the point F, which will be the centre. Then a perpendicular from F, drawn upon any one of the faces will be the radius of the inscribed sphere.

16. For the inclination of any two contiguous faces of the octahedron, see Theorem 34, page 376.

The octahedron is divisible into two pyramids whose bases are squares, and the four slant sides equilateral triangles.

17. Every oblique pyramid may be proved to be equal to a right pyramid of the same base and altitude.

Every right pyramid whose base is not triangular may be divided into triangular pyramids of the same altitude.

Every pyramid on a triangular base may be proved equal to one-third of a prism of the same base and altitude.

Hence, any pyramid may be proved to be one-third of a prism of the same base and altitude.

18. If the centres of the upper sphere, and the three upon which it rests, be joined, the figure is a tetrahedron: and the same remark may be made with respect to each of the three, and the spheres upon which they severally rest.

## HINTS, &c. TO THE THEOREMS. BOOK XII.

5. Let a regular polygon be inscribed in the circle. The straight lines drawn from the centre to all the angles of the polygon, will divide the polygon into as many equal isosceles triangles, as there are sides of the polygon, and perpendiculars drawn from the centre on each side, will be equal to the common altitude of all the triangles. Each of these triangles is equal to half the rectangle contained by the base and altitude of the triangle. Hence the area of the polygon is equal to half the rectangle contained by the common altitude and the sum of the sides of the polygon. Now if the sides of the regular polygon be diminished in magnitude and their number increased, the perimeter of the polygon may be made continually to approach to the perimeter of the circle, and at length be made to differ from it by a magnitude less than can be assigned. In that case also the perpendiculars on the sides of the polygons differ from the radius of the circle by a length less than can be assigned. Hence the area of the polygon and the area of the circle differ from each other by a quantity less than can be assigned, and therefore the area of the circle is equal to half of the rectangle contained by two straight lines which are equal to the radius and the circumference of the circle: or the area of the circle is equal to the rectangle contained by the radius and a straight line equal to half the circumference of the circle.

6. The angle in a segment which is one-fourth of the circumference of a circle, is equal to one of the interior angles of a regular octagon. The ratio of the two angles will be found to be as 3 to 2.

7. Let AB, A′B′ be arcs of concentric circles whose centre is C and radii CA, CA′, and such that the sector ACB is equal to the sector A′CB′. Assuming that the area of a sector is equal to half the rectangle contained by the radius and the included arc: the arc AB is to the arc A′B′ as the radius A′C is to the radius AC. Let the radii AC, BC be cut by the interior circle in A′, D. Then the arc A′D is to the arc AB, as A′C is to AC; because the sectors A′CD, ACB are similar: and the arc AB′ is to the arc AD, as the angle ACB′ is to the angle ACD, or the angle ACB. Euc. VI. 33.

From these proportions may be deduced the proportion:—as the angle ACB is to the angle A′CB′, so is the square of the radius A′C to the square of the radius AC.

And by Euc. XII. 2, the property is manifest.

8. Let AB, A′B′ be arcs of two concentric circles, whose centre is C. ACB, A′CB′ two sectors such that the angle ACB is to the angle A′CB′, as $A'C^2$ is to $AC^2$.

Let AC, BC be cut by the interior circle in A′, D;

Then the arc A′B′ is to the arc A′D, as the angle A′CB′ is to the angle A′CD, or the angle ACB. Euc. VI. 33.

And the arc A′D is to the arc AB, as the radius A′C is to the radius AC, by similar sectors.

By means of these two proportions and the given proportion, the rectangle con-

tained by the arc AB and radius AC, may be proved equal to the rectangle contained by the arc A'B' and the radius A'C.

9. The sum of the squares of the segments of the diagonals, is equal to the sum of the squares of each pair of opposite sides of the quadrilateral figure. Hence by Euc. XII. 2; I. 47; V. 18, the property is proved.

10. The radii of the circles may be proved to be proportional to the two sides of the original triangle. Then by Euc. XII. 2; VI. 19.

11. Let the two lines intersect each other in A, and let C be the centre of the last circle. Join CA, and draw CB perpendicular to AB one of the lines.

Then CA and CB are given. Let C', C'', &c., be the centres of the circles which successively touch one another. Draw C'B', C''B'', &c., perpendicular to AB and C'D, C''D' parallel to AB meeting CB in D, C'B' in D', &c. Then by means of the similar triangles, the radii C'B', C''B'', &c., may be expressed in terms of CB and CA. Hence the sum of the series of the circles may be expressed in terms of the area of the last circle.

12. The squares of the four segments, are together equal to the square of the diameter. Theorem 4, p. 314; and Theorem 129, p. 325. Then by Euc. XII. 2; V. 18, the truth of the Theorem is manifest.

13. This is shewn by Euc. I. 47; XII. 2; V. 18.

14. The demonstration of this property is contained in that of Problem 5, p. 374.

15. Let C*c* be the line joining the centres of the two circles whose planes are parallel, and let ACB, *acb* be parallel diameters drawn in each. Join AC, B*b*, then AB*ba* is a quadrilateral figure having two of its sides AB, *ab* parallel. It is then required to shew that the lines joining A*b*, B*a* intersect at the same point D in the line C*c*. If *ab* be equal to AB, the figure AB*ba* is a rectangular parallelogram.

16. Let A be any point above the plane of the circle whose centre is C and diameter BCD. Join CA, and let a plane pass through any point *c* in AC or AC produced. Through *c* in this plane draw *bcd* parallel to BCD. Join BA, DA and produce them to meet *bcd* in *d* and *b*. Then *b*, *d* may be proved to be two points in the circumference of the circle whose diameter is *bcd*, and by means of the similar triangles A*cd*, ACD, the areas of the two circles may be shewn to be proportional to the squares of AC and A*c*.

17. Let the arc of a semicircle on the diameter AB be trisected in the points D, E; C being the centre of the circle. Let AD, AE, CD, CE be joined, then the difference of the segments on AD and AE, may be proved to be equal to the sector ACD or DCE.

18. The proof of this property depends on the demonstration of Theorem 2, p. 371, and the relation between the area of two circles described upon two lines as diameters, one of which is double of the other.

19. Let the figure be described, and the demonstration will be obvious from the consideration of the parts.

20. The triangles ABD, ABC have the angle ABD common, and ACB may be proved equal to BAD, by Euc. I. 29; III. 32. And therefore the angle CAB is equal to the angle ADB. Also the triangles CAB, CEA may be shewn to be equiangular, and AD equal to AE. Then by Euc. VI. 4.

21. Let the diameter AB be divided into five equal parts, in C, D, E, then C, D are the second and third points of division. The semicircles AEC, AFD are described on one side of the diameter, and BGC, BHD on the other. Then since the semicircumferences of circles are proportional to their diameters, the perimeter of the figure AECGBHDF is shewn to be equal to the perimeter of the original circle.

By Euc. XII. 2, the area of the figure AECGBHD may be shewn to be one fifth part of the area of the circle.

The general case, when the diameter is divided into *n* equal parts is proved in the same way.

22. This is shewn from Euc. XII. 2; I. 47; V. 18.

23. Assuming that the area of a sector of a circle is equal to half the rectangle contained by the radius and the arc, the sector AOC is shewn to be equal to the triangle AOB.

24. By Euc. XII. 2, the area of the quadrant ADBEA is equal to the area of the semicircle ABCA.

25. Let POQ be any quadrant, O being the centre of the circle, and let BG, DH be drawn perpendicular to the radius PO, and OB, OD be joined. The triangle GBO is equal to DHO.

26. The segments on BC, BA, AC may be shewn to be similar. And similar segments of circles may be proved to be proportional to the squares of their radii, Euc. XII. 2, and to the squares of the chords on which they stand, Euc. VI. 6.

If Euc. VI. 31 be extended to *any similar figures*, the equality follows directly.

27. The triangles CEA, CEB are equal, and the difference of the two segments may be shewn to be equal to the difference of the parts of the semicircle made by CE. The difference of the same parts may also be shewn to be equal to double the sector DEC.

28. Let AB be the hypothenuse of the right-angled triangle ABC, and let the semicircles described upon the sides AC, BC, intersect the hypothenuse in D. Join AD. AD is perpendicular to AB. The segments on AC, AD, and on one side of CD are similar; and the segments on AC may be shewn to be equal to the segments on AD, CD. Also the segment on BC may be shewn to be equal to the segments on BD and the other side of CD.

If however Euc. VI. 31 be true for *all similar figures*, the conclusions above stated follow at once from the right-angled triangles.

29. (*a*) Join BD, CD, DA, Euc. III. 31; I. 14. (*b*) Produce CD to meet the arc of the quadrant in E. Then the sector ACE is half of the quadrant: also the semicircle CDA may be shewn to be equal to half the quadrant. (*c*) The segments on CD and DA are similar and equal, if the figure bounded by DA, AC, and the arc CD be added to each, the remaining part of the semicircle on AC is equal to the triangle ACD which is a right-angled isosceles triangle.

30. This theorem is analogous to Euc. III. 14.

31. Let D be the given point, and from D let DA be drawn through the centre E, and meeting the surface in C, A. Let DB be a line from D touching the sphere at B. Join BE. Then the triangle DBE (fig. Euc. III. 36) is in a plane passing through D, and E the centre of the sphere, and the distances DE, EB are always the same. Hence it follows that BD is always of the same length. Euc. I. 47.

The sphere which touches the six edges of any tetrahedron, has four circular sections touching the sides of the four triangles which form the surface of the tetrahedron.

32. Let the circle ADB cut the circle AEB in the diameter AB at any angle, C being their common centre. Next let the plane perpendicular to AB cut the circumference of the circle ADB in D, F, and the circumference of AEB in E, G. Then E, D, G, F may be proved to be in the circumference of a circle.

33. See the Géométrie par M. Vincent, p. 450.

34. Let ABCD be a regular tetrahedron. From A in the plane ABC draw AE perpendicular to BC, and join DE in the plane BCD, also from A draw AG perpendicular to the line DE. Then the angle AEG is the inclination of the two faces ABC, DBC of the tetrahedron, and the base EG is one-third of the hypothenuse AE in the right-angled triangle AGE.

Let $abcdef$ be a regular octahedron whose faces are equal to those of the tetrahedron. Join $a$, $f$ two opposite vertices. Draw $ag$ in the plane $abc$ perpendicular to $bc$, and $ge$ perpendicular to $af$. Draw $fg$ in the plane $fbc$, and from $f$ draw $fh$ perpendicular to $ag$ produced.

Then $agf$ is the inclination of two faces of the octahedron. Also in the right-angled triangle $fhg$, $gh$ may be proved to be one-third of $fg$, and $fg$ is equal to AE. Hence the triangles $fgh$, AEF are equal in all respects. Therefore the angle $fgh$ is equal to the angle AEB. Hence the angle AEF is the supplement of the angle $agf$, or the inclination of two contiguous faces of a tetrahedron, is the supplement of the inclination of two contiguous faces of an octahedron.

# ADDENDUM.

## PROBLEM 16, page 333.

Inscribe in a circle a triangle whose sides or sides produced, shall pass through three given points in the same plane.

Lemma I. Let there be given two points A, B, and a circle DCE whose centre is S: from the point A, draw ADC to cut the circle; through B, C, D describe a circle cutting the line AB in K: then K is a fixed point, however the line ADC may be drawn; and if KD be drawn to meet the circle in H, the line HE, drawn to the intersection E of BC with the circle, will be parallel to AB.

For, first, BA . AK = CD . DA = a given magnitude, namely, the square of the tangent from A to the circle. Also BA is given, and hence AK is also given, and K is a fixed point, however ADC be drawn from A to cut the circle.

And, secondly, since CDHE is a quadrilateral inscribed in the circle CDE, the exterior angle made by producing EH, is equal to the interior opposite angle DCE. In the same way the angle DKA is equal to DCB. Whence these angles are equal, and HE is parallel to AB.

Lemma II. The same conditions as before being given, draw the diameter TV through K and S; and find the point F such that SF . SK = $ST^2$: then joining FD, the angle EDF will be equal to the difference between BKS and a right angle, however the point C be taken in the circle DCE.

For, draw HX parallel to KS, and from X draw the diameter XSG, and join HG, GD; also draw KP perpendicular to AB.

Then since XHG is an angle in a semicircle, it is a right angle; and since HX is parallel to KS, HG is perpendicular to KS; and the angle EHG is equal to PKS.

Moreover, since HG is perpendicular to KS, the line DG always passes through F, and hence the line FG makes with DE the constant angle PKS, however C may be taken in the circle DCE.

Having premised these two Lemmas, we may proceed to the construction of the Problem, as follows:

Let A, B, Q be the three given points. Find the points K, F as in the Lemmas, together with the angle PKS. Join QF, and on it describe a segment to contain the angle PKS; and let it cut the circle DCE in D. Then D is one of the angular points of the triangle. Join DA meeting the circle in C, and CB meeting the circle in E, and draw ED. It will pass through Q by the reasoning of the Lemmas.

The same principle may be applied to the solution of Problem 58, p. 324.

Any two points, A, B, being given within a circle CDE, it is required to find a point D, so that the difference of the angles BDE, ADC may equal a given angle.

The points A, B may be taken any where either within or without the circle; and the construction will be the same.

Analysis. Let the point D be supposed to be found, such that the angle EDB − DCA = the given angle. Make ECL equal to that angle, and join LC. Then, obviously, if BD meet the circle in H, and DA meet it in G, the chord GH will be parallel to LC. Find F and K, by the Lemmas, then the angle FHG is given. Wherefore through F draw FQ parallel to LC, and make the angle HFG equal to the given angle. Draw BH meeting the circle in D, and join DA: then these are the lines required.

NOTE.—The second Lemma is only a variation of the last Porism of Euclid's third Book on that subject.

The 57th proposition in Dr Simson's Restoration of the Porisms, leads directly to the construction in the manner here given.

The former Problem, though not mentioned directly by Pappus, nor found in any ancient author, was without doubt considered by the Greek geometers.

It has been regarded by modern geometers as an extension of the 117th Proposition of the Seventh Book of the Collections of Pappus, namely:—when the three points are not in the same straight line.

The Problem itself, as well as Proposition 117 of Pappus, has engaged the attention of several distinguished modern geometers. Bonnycastle, in p. 348 of his Geometry, has given a concise account of the several solutions by mathematicians on the continent; as also Dr Traill in p. 95 of his life of Simson. In p. 97, he has given Simson's solution of the Problem, which, from a note attached, appears to have been completed in 1731. Simson's "Opera Reliqua" was published by the munificence of Earl Stanhope in 1776.

In 1742, M. Cramer proposed the problem to M. de Castillon, but it was not till 1776 that Castillon published a geometrical solution in the Berlin Memoirs of that year. In the same volume is a solution by La Grange, by means of trigonometrical formulæ. Carnot, in p. 383 of his *Géométrie de Position*, has given a modified form of La Grange's Solution. In the Petersburgh Acts for 1780, are solutions of the same Problem by Euler, Lexell and Fuss. In the Memoirs of the Italian Society (Tom. IV. 1788) are two papers respecting this problem; one by Ottajano, in which is given a geometrical solution of the problem, and an extension to the case of a polygon of any number of sides, which he inscribes in a given circle, so that the sides respectively shall pass through the same number of points. Ottajano also gives a sketch of the history of the problem. The other paper is by Malfatti, and contains a solution of the general problem of the polygon last mentioned.

In the Berlin Memoirs for 1796 is a paper by Lhuilier, containing an algebraical solution of the most general case of the polygon. He also extends the problem to the conic sections, and adds a similar one respecting the sphere. An extension of this Problem to the Conic Sections has also been effected by Poncelet in his *Traité des Propriétés Projectives*. M. Brianchon has considered the Problem in the case where a conic section is substituted for the circle, and where the three points are in one line. His solution will be found in the Journal de l'Ecole Polytechnique. (Tom. IV.)

Dr Wallace and Mr Lowry applied the 57th Porism to this problem in Vol. II. of the Old Series of Leybourn's Repository. In Vol. I. of the New Series of that work, a new and very elegant analysis of the Porism was given by Mr Noble, which has been the main guide in demonstrating the two Lemmas here used. The same method, slightly modified, applies to any inscribed polygon. The most elegant system of investigation, however, that has ever been published, is that of Mr Swale, in the second number of his Apollonius. Mr Lowry has also given the solution of the problem in the case where the ellipse is substituted for the circle, and where the polygon has any number of sides. See Leybourn's Repository, Vol. II., New Series, p. 189.

# SIMSON'S RESTORATION OF THE PORISMS.

It is proposed to publish, by subscription, a translation of Dr Simson's Restoration of the Porisms. The subject forms an interesting portion of the Ancient Greek Geometry, and the extreme scarcity of Simson's Opera Reliqua, may, perhaps, appear to be some apology for the undertaking.

The Translation will be preceded by a full discussion of the peculiar character of these propositions, and an exposition of the sources from which many mistakes respecting them have flowed. Occasional notes will be added on particular propositions; and a full development of the Algebraical method of investigating the Porisms will be affixed to the Translation.

The Treatise on Porisms in the Opera Reliqua occupies 278 quarto pages, and from the probable number of subscribers, which may be expected, it is calculated that the price of each copy of the work will not exceed ten shillings.

As soon as a number of subscribers sufficient to defray the expenses of printing, &c., has been obtained, it is proposed to put the work to press.

The work will be printed at the University Press, in octavo.

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